Function Spaces, Differential Operators and Nonlinear Analysis

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Besov-type spaces of variable smoothness on rough domains

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Preliminaries

Let us introduce the class of admissible weight sequences.

Definition 1 ([1], [2]) Let $\{s_k\} = \{s_k(\cdot)\}_{k=0}^{\infty}$ be a sequence of strictly positive weights. Let $\alpha_3 \geqslant 0$, $\alpha_1, \alpha_2 \in \mathbb{R}$. A sequence $\{s_k\}$ will be said to lie in $\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ if for every $x, y \in \mathbb{R}^n$

$$\begin{aligned} &1) \qquad \frac{1}{C_1} 2^{\alpha_1(k-l)} \leqslant \frac{s_k(x)}{s_l(x)} \leqslant C_1 2^{\alpha_2(k-l)}, \quad l \leqslant k \in \mathbb{N}_0; \\ &2) \qquad s_k(x) \leqslant C_2 s_k(y)(1 + 2^k|x-y|)^{\alpha_3}, \quad k \in \mathbb{N}_0. \end{aligned} \qquad (1)$$

The following definition was firstly introduced (for $p, q \in (1, \infty)$) by O.V. Besov [1] and then extended by H. Kempka [2].

Definition 2 ([2]) Let $p, q \in (0, \infty]$ and $\{s_k\} \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. Then

$$B^{\{s_k\}}_{p,q}(\mathbb{R}^n) := \{f \in S'(\mathbb{R}^n) : \|f|B^{\{s_k\}}_{p,q}(\mathbb{R}^n)\| < \infty\},$$
$$\|f|B^{\{s_k\}}_{p,q}(\mathbb{R}^n)\| := \left(\sum_{k=0}^{\infty} \|s_k F^{-1}\Psi_k Ff|L_p(\mathbb{R}^n)\|^q\right)^{\frac{1}{q}}, \tag{2}$$

where $\{\Psi_k\}_{k=0}^{\infty}$ is the standard resolution of unity.

Spaces on open sets

Let U be an open subset of $\mathbb{R}^n$ and $p, q \in (0, \infty]$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_3 \geqslant 0$ and let $\{s_k\} \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ be a weight sequence. The space $B^{\{s_k\}}_{p,q}(U)$ is defined to be the restriction of the corresponding space from $\mathbb{R}^n$ to U. This space is endowed with the quotient space quasi-norm. More precisely, for $f \in D'(U)$,

$$\|f|B^{\{s_k\}}_{p,q}(U)\| = \inf\{\|g|B^{\{s_k\}}_{p,q}(\mathbb{R}^n)\| : g|_U = f \text{ in } D'(U)\}.$$

Statement of the Problem

Problem A

Find an intrinsic and constructive description of the space $B^{\{s_k\}}_{p,q}(U)$. More precisely, it is required to find equivalent norm in the space $B^{\{s_k\}}_{p,q}(U)$ which would utilize only the information about the distribution (function) on an open set U.

Problem B

Construct a bounded linear operator $\mathrm{Ext} : B^{\{s_k\}}_{p,q}(U) \to B^{\{s_k\}}_{p,q}(\mathbb{R}^n)$ which is right inverse for the operator Tr.

History of the Problem

For classical Besov $B^s_{p,q}(\mathbb{R}^n)$ and Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$ the Problems A and B was solved by V. Rychkov in 2000 [3]. It was assumed that U is either bounded or special Lipschitz domain.

Despite of the vast literature devoted to spaces of variable smoothness, the question of intrinsic description of traces of such spaces on rough domains remained open.

Our approach

We will obtain the solutions of the Problems A and B as a particular case of more general problems.
Namely, we introduce much more general Besov-type space of variable smoothness and solve our Problems A and B for such space.

New Besov-type space of variable smoothness

Definition 1 is not satisfactory for the following reasons:
1) the weight sequence $\{s_k\}$ contains functions that grow slowly at infinity,

2) every function s_k has not any singularities,

3) the numbers $\alpha_1, \alpha_2, \alpha_3$ from the Definition 1 can measure only local pointwise oscillations.

New weight class

Definition 3 Let $p, \sigma_1, \sigma_2 \in (0, \infty]$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_3 \geqslant 0$, and let $\sigma := (\sigma_1, \sigma_2)$, $\alpha := (\alpha_1, \alpha_2)$. We say that $\{t_k\} := \{t_k(\cdot)\}_{k=0}^{\infty} \in \mathcal{X}^{\alpha_3}_{\alpha,\sigma,p}$ iff for $C_1, C_2 > 0$ and for all $m \in \mathbb{Z}^n$

1)

$$\Big(2^{kn} \int\limits_{Q_{k,m}} t_k^p(x)\Big)^{\frac{1}{p}} \Big(2^{kn} \int\limits_{Q_{k,m}} t_j^{-\sigma_1}(x)\Big)^{\frac{1}{\sigma_1}} \leqslant C_1 2^{\alpha_1(k-j)}, \qquad 0 \leqslant k \leqslant j,$$

2)

$$\Big(2^{kn} \int\limits_{Q_{k,m}} t_k^p(x)\Big)^{-\frac{1}{p}} \Big(2^{kn} \int\limits_{Q_{k,m}} t_j^{\sigma_2}(x)\Big)^{\frac{1}{\sigma_2}} \leqslant C_2 2^{\alpha_2(j-k)}, \qquad 0 \leqslant k \leqslant j,$$

3)

$$\int\limits_{Q_{k,m}} t_k^p(x)\, dx \leqslant 2^{\alpha_3} \int\limits_{Q_{k,\widetilde{m}}} t_k^p(x)\, dx, \qquad |m - \widetilde{m}| \leqslant 1, \quad k \in \mathbb{N}_0.$$

New weight class

Properties of the weight class $\mathcal{X}^{\alpha_3}_{\alpha,\sigma,p}$:
1) $\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2} \subset \mathcal{X}^{\alpha_3}_{\alpha,\sigma,p}$ and inclusion is strict if $\sigma_1, \sigma_2 < \infty$;
2) $\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2} = \mathcal{X}^{\alpha_3}_{\alpha,\sigma,p}$ if and only if $\sigma_1 = \sigma_2 = \infty$;
3) if $\gamma \in A_q(\mathbb{R}^n)$ and $\{s_k\} \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ then $\{\gamma s_k\} \in \mathcal{X}^{\alpha_3}_{\alpha,\sigma,p}$ for some $\sigma_1(q), \sigma_2(q)$.

From the property 3) it follows that the multiplication of a fairly 'good' sequence $\{s_k\}$ by a sufficiently 'bad' weight γ does not impair the exponents α_1, α_2.

Following [3], [4] by S_e we denote the set of all $f \in C^\infty$ such that

$$p_N(f) := \sup_{x\in\mathbb{R}^n} \exp(N|x|) \sum_{|\alpha|\leqslant N} |D^\alpha f(x)| < \infty \quad \text{for all } N \in \mathbb{N}_0. \tag{3}$$

We equip S_e with the locally convex topology defined by the system of the semi-norms p_N.
By S_e' we denote the collections of all continuous linear forms on S_e. We equip S_e' with the strong topology (see [4] for details).

New Besov-type space of variable smoothness

Let $p, q, \sigma_1, \sigma_2 \in (0, \infty]$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_3 \geqslant 0$, and let $\{t_k\} \in \mathcal{X}^{\alpha_3}_{\alpha,\sigma,p}$ be a weight sequence. Consider $\varphi_0 \in D$ such that $\int \varphi_0(x)\, dx = 1$. Next, we set $\varphi(x) := \varphi_0(x) - 2^{-n}\varphi_0(\frac{x}{2})$, where $x \in \mathbb{R}^n$.

Definition 4 (T. [5]) By $\mathfrak{B}^{\{t_k\}}_{p,q}(\mathbb{R}^n) := \mathfrak{B}^{\{t_k\}}_{p,q}(\varphi_0, \mathbb{R}^n)$ we shall denote the set of all distributions $f \in S'_e$ with finite quasi-norm

$$\|f|\mathfrak{B}^{\{t_k\}}_{p,q}(\mathbb{R}^n)\| := \left(\sum_{k=0}^{\infty} \|t_k(\varphi_k * f)|L_p(\mathbb{R}^n)\|^q\right)^{\frac{1}{q}}. \qquad (4)$$

Elementary properties of the space $\mathfrak{B}^{\{t_k\}}_{p,q}(\mathbb{R}^n)$

Let $p, q, \sigma_1, \sigma_2 \in (0, \infty]$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_3 \geqslant 0$, and let $\{t_k\} \in \mathcal{X}^{\alpha_3}_{\alpha,\sigma,p}$. Then (T. [5], [7])

1) $\mathfrak{B}^{\{t_k\}}_{p,q}(\varphi_0, \mathbb{R}^n) \subset S'_e(\mathbb{R}^n)$ and embedding is continuous;

2) the space $\mathfrak{B}^{\{t_k\}}_{p,q}(\varphi_0, \mathbb{R}^n)$ is complete;

3) if $1 + L_\varphi > \alpha_2$ and $\sigma_2 \geqslant p$ then the space $\mathfrak{B}^{\{t_k\}}_{p,q}(\mathbb{R}^n) := \mathfrak{B}^{\{t_k\}}_{p,q}(\varphi_0, \mathbb{R}^n)$ is independent of the choice of the function φ_0 and the corresponding quasi-norms are equivalent;

4) if $\{t_k\} \in \mathcal{W}^{\beta_3}_{\beta_1,\beta_2}$ and $1 + L_\varphi > \beta_2$ then $\mathfrak{B}^{\{t_k\}}_{p,q}(\varphi_0, \mathbb{R}^n) = B^{\{t_k\}}_{p,q}(\mathbb{R}^n)$ and corresponding norms are equivalent.

Main Results

Theorem 1 (pointwise multipliers) Let $\varphi_0 \in D$, $\int \varphi_0(x)\,dx = 1$ and $\varphi := \varphi_0 - 2^{-n}\varphi_0(\frac{\cdot}{2})$. Let $\sigma_2 \geqslant p$ and $L_\varphi + 1 > \alpha_2$, $r > \alpha_2$. Then, for all $f \in \mathfrak{B}^{\{t_k\}}_{p,q}(\mathbb{R}^n)$ and $\omega \in C^r_0(\mathbb{R}^n)$ we have $\omega f \in \mathfrak{B}^{\{t_k\}}_{p,q}(\mathbb{R}^n)$ and

$$\|\omega f|\mathfrak{B}^{\{t_k\}}_{p,q}(\mathbb{R}^n)\| \leqslant C\|\omega|C^r_0(\mathbb{R}^n)\|\|f|\mathfrak{B}^{\{t_k\}}_{p,q}(\mathbb{R}^n)\|. \tag{5}$$

This result generalizes corresponding result from [6] (in the case of constant p, q) to the case of more general weight sequence.

Main Results

Let us recall definitions of bounded Lipschitz and special Lipschitz domains.
1) A bounded Lipschitz domain is a bounded domain G, whose boundary ∂G can be covered by a finite number of balls B_k so that, possibly after a proper rotation, $\partial G \bigcap B_k$ for each k is a part of the graph of a Lipschitz function.
2) A special Lipschitz domain is defined as an open set G lying above the graph of a Lipschitz function.

Main Results

Now we can give a solution of the Problem A for the space $\mathfrak{B}^{\{t_k\}}_{p,q}(\mathbb{R}^n)$ (and hence for the space $B^{\{t_k\}}_{p,q}(\mathbb{R}^n)$ in the case $\{t_k\} \in \mathcal{W}^{\beta_3}_{\beta_1,\beta_2}$) for bounded Lipschitz or special Lipschitz domains. By virtue of Theorem 1 we may restrict ourselves to the case special Lipschitz domain.

Theorem 2 (solution of the Problem A)
Let $p, q \in (0, \infty]$. Let $\varphi_0 \in D(-K)$, $\int \varphi_0(x)\, dx = 1$ and $\varphi := \varphi_0 - 2^{-n}\varphi_0(\frac{\cdot}{2})$. Let $\sigma_2 \geqslant p$ and $L_\varphi + 1 > \alpha_2$. Let G be a special Lipschitz domain. Then, for all $f \in S'_e(G)$,

$$\|f|\mathfrak{B}^{\{t_k\}}_{p,q}(G)\| \approx \Big(\sum_{j=0}^{\infty}\Big(\int_G t_j^p(x)|\varphi_j * f(x)|^p\, dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}. \qquad (6)$$

Main Results

Here we present our solution of the Problem B.

Theorem 3 (solution of the Problem B) Let $\varphi_0 \in D(-K)$, $\int \varphi_0(x)\,dx = 1$ and $\varphi := \varphi_0 - 2^{-n}\varphi_0(\frac{\cdot}{2})$. Let $\sigma_2 \geqslant p$, $L_\varphi + 1 > \alpha_2$, $1 + L_\psi > \max\{0, -\alpha_1\}$. Then the map $\mathrm{Ext} : D'(G) \to D'(\mathbb{R}^n)$, as defined by

$$\mathrm{Ext}[f] = \sum_{j=0}^{\infty} \psi_j * (\varphi_j * f)_G \tag{7}$$

defines a linear bounded operator from the space $\mathfrak{B}^{\{t_k\}}_{p,q}(G)$ into the space $\mathfrak{B}^{\{t_k\}}_{p,q}(\mathbb{R}^n)$.

Main Results

There is an alternative (but nonequivalent) approach to Besov-type spaces with variable smoothness $\{t_k\} \in \mathcal{X}^{\alpha_3}_{\alpha,\sigma,p}$ which is based on local polynomial approximation technique (T. [8]).

Such approach gives us solution of Problems A and B for (ε, δ) domains (T. [7]).

Thank you for your attention

THANK YOU FOR YOUR ATTENTION !

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