On anisotropic smoothness of solutions to elliptic equations in domains with continuous boundary.

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The objects and a problem.

Let

1. (M, g) *be a smooth* ***compact*** *Riemannian manifold (possibly) with boundary.*
2. $\mathcal{A}$ *be an elliptic differential operator of order* 2*, generated by the following expression*

$$\mathcal{A}u = -\mathrm{div}(A\nabla u) + b\nabla u + cu$$

3. Ω *be a subdomain (of* M*) with* ***continuous boundary*** $\partial\Omega \in C^{0,\omega(\cdot)}$

Consider

Dirichlet problem

$$\mathcal{A}u = f, \quad f \in H^{-1}(\Omega), \tag{1}$$

solution $u \in \mathring{H}^1(\Omega)$ *is understood in the weak* ***variational*** *sence.*

We are interested in:

What one can say about smoothness of solution u? Is it possbile to ensure that $u \in \tilde{H}^{1+\varepsilon_2}(\Omega)$ when $f \in H^{-1+\varepsilon_1}(\Omega)$. How does the operator and the boundary $\partial\Omega$ impact on the value of ε_2.

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Known results.

Theorem (Nirenberg, 1955 / Lions–Magenes, 1972)

Let

1. Ω *be a bounded domain of* $\mathbb{R}^d$, $\partial\Omega \in C^{1,1}$ *or* Ω — *convex*
2. $\mathcal{A} = -\Delta$,
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Then solution of (1) belongs to $H^2(\Omega) \cap \mathring{H}^1(\Omega)$.

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$$\mathcal{G}_\Omega : H^{-1}(\Omega) \to \tilde{H}^1(\Omega)$$
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Let $\partial\Omega \in C^{0,\gamma}$, $\Omega \Subset M$, $\gamma \in (0,1]$, $t \in (0,1)$ and

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Let X be a Banach space of functions, $\Delta_h u = u_h - u$, $u_h(x) = u(x+h)$, $\overline{\omega}$ be a vector of moduli continuity, $\{e_1, e_2, \ldots, e_d\}$ be a basis of $\mathbb{R}^d$, $q \in [1, \infty]$. Then

$$B_q^{\overline{\omega}}(X) \stackrel{\text{def}}{=} \left\{ u \in X \;\middle|\; \max_i \left\| \frac{\|\Delta_{he_i} u\|_X}{\omega_i(|h|)} \right\|_{L_q^*(\mathbb{R})} < \infty \right\},$$

where $\|f(h)\|_{L_q^*(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(h)|^q \frac{dh}{|h|} \right)^{1/q}$, $q \in [1, \infty)$ and $\|f(h)\|_{L_\infty^*(\mathbb{R})} = \|f(h)\|_{L_\infty(\mathbb{R})} = \operatorname{ess\,sup} |f(h)|$. For $k \in \mathbb{Z}_+$ we denote

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Nikol'skii – Besov spaces.

Let us define K–spaces with negative smoothness on domains. For this aim consider the following spaces

$$\mathring{\tilde{K}}_p^{k,\overline{\omega}}(\Omega) = [C_0^\infty(\Omega)]_{K_p^{k,\overline{\omega}}(\mathbb{R}^d)}$$

then for $k \in \mathbb{N}$ we denote by

$$K_{p'}^{-k,\overline{\omega}}(\Omega) = \left(\mathring{\tilde{K}}_p^{k-1,\frac{(\cdot)}{\overline{\omega}}}(\Omega)\right)^\star, \qquad 1/p + 1/p' = 1.$$

Since on manifolds we have no linear structure, we should intrdouce some athlas $\mathcal{V}$ w.r.t. we can measure anisotropic smoothness. Hence, we will use the following notation $\left(K_{p'}^{-k,\overline{\omega}}(\Omega)\right)_{\mathcal{V}}$.

Pointwise multipliers to negative smoothness.

Let us introduce spaces of pointwise multipliers.

Let X, Y be functional Banach spaces such that $C_0^\infty(\Omega)$ densely imbedded to X and Y. Then the space

$$M(X \to Y^\star) = M(Y \to X^\star)$$

consists of all functions u, such that the following estimate holds

$$\sup_{\substack{v,w \in C_0^\infty(\Omega), \\ \|v\|_X = 1 \\ \|w\|_Y = 1}} \frac{\int_\Omega |u \cdot v \cdot w| \mathrm{d}Vol}{\|v\|_X \|w\|_Y} = \|u\|_{M(X \to Y^\star)} < \infty.$$

One more thing. For moduli continuity ω_1, ω_2 we say that $\omega_1 \preceq \omega_2$, if there exists such constants C that $\omega_2(x) \leq \mathrm{C}\omega_1(x)$ holds for all $x \in [0, 1]$. For instance $h^{\gamma_1} \preceq h^{\gamma_2}$, if $\gamma_1 \leq \gamma_2$.

Theorem

Let $\partial\Omega \in C^{0,\overline{\omega}}_{\mathcal{V}}$ *and*

A1. $\forall x \in M$, $\forall \xi, \eta \in \otimes_{\mathbb{C}} T^*_x M \Rightarrow \boldsymbol{A}(\xi \otimes \bar{\eta}) = \overline{\boldsymbol{A}(\eta \otimes \bar{\xi})}$;

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A3. $\boldsymbol{A} \in C^{0,\overline{\varkappa}}_{\mathcal{V}}(M)$, $\overline{\varkappa} = (\varkappa_1, \ldots, \varkappa_d)$, $\varkappa_k$ *be moduli continuity.*

A4. $b \in M\left(L_2(\Omega) \to \left(K_2^{-1,\overline{\gamma}}\right)_{\mathcal{V}}(\Omega)\right)$, $c \in M\left(\tilde{H}^1(\Omega) \to \left(K_2^{-1,\overline{\gamma}}\right)_{\mathcal{V}}(\Omega)\right)$.

Then the solving operator $\mathcal{R}$ *is bounded:*

$$\mathcal{R} : \left(B_{2,1}^{-1,\bar{\epsilon}}\right)_{\mathcal{V}}(\Omega) \to \left(\tilde{N}_2^{1,\bar{\epsilon}}\right)_{\mathcal{V}}(\Omega), \quad \bar{\epsilon} = (\epsilon_1, \ldots, \epsilon_d),$$

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A3. $\boldsymbol{A} \in C^{0,\overline{\varkappa}}_{\mathcal{V}}(M)$, $\overline{\varkappa} = (\varkappa_1, \ldots, \varkappa_d)$, $\varkappa_k$ *be moduli continuity.*

A4. $b \in M\left(L_2(\Omega) \to \left(K_2^{-1,\overline{\gamma}}\right)_{\mathcal{V}}(\Omega)\right)$, $c \in M\left(\tilde{H}^1(\Omega) \to \left(K_2^{-1,\overline{\gamma}}\right)_{\mathcal{V}}(\Omega)\right)$.

Then the solving operator $\mathcal{R}$ *is bounded:*

$$\mathcal{R} : \left(B_{2,1}^{-1,\overline{\epsilon}}\right)_{\mathcal{V}}(\Omega) \to \left(\tilde{N}_2^{1,\overline{\epsilon}}\right)_{\mathcal{V}}(\Omega), \quad \overline{\epsilon} = (\epsilon_1, \ldots, \epsilon_d),$$

where

- $\epsilon_i \preceq \sqrt{\varkappa_i}$,
- $\exists c \in (0,1) : \epsilon_i \preceq \gamma_i^c$, $i = \overline{1, \ldots, d}$,
- $\epsilon_j = \epsilon_d \circ \omega_j$, $j = \overline{1, \ldots, d-1}$.

Theorem

Let $\partial\Omega \in C^{0,\overline{\omega}}_{\mathcal{V}}$ *and*

A1. $\forall x \in M$, $\forall \xi, \eta \in \otimes_{\mathbb{C}} T^*_x M \Rightarrow \boldsymbol{A}(\xi \otimes \bar{\eta}) = \overline{\boldsymbol{A}(\eta \otimes \bar{\xi})}$;

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Then the solving operator $\mathcal{R}$ *is bounded:*

$$\mathcal{R} : \left(B^{-1,\overline{\epsilon}}_{2,1}\right)_{\mathcal{V}}(\Omega) \to \left(\tilde{N}^{1,\overline{\epsilon}}_2\right)_{\mathcal{V}}(\Omega), \quad \overline{\epsilon} = (\epsilon_1, \ldots, \epsilon_d),$$

where

- $\epsilon_i \preceq \sqrt{\varkappa_i}$,
- $\exists c \in (0,1) : \epsilon_i \preceq \gamma_i^c$, $i = \overline{1, \ldots, d}$,
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Theorem

Let $\partial\Omega \in C^{0,\overline{\omega}}_{\mathcal{V}}$ *and*

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A3. $\boldsymbol{A} \in C^{0,\overline{\varkappa}}_{\mathcal{V}}(M)$, $\overline{\varkappa} = (\varkappa_1, \ldots, \varkappa_d)$, $\varkappa_k$ *be moduli continuity.*

A4. $b \in M\left(L_2(\Omega) \to \left(K_2^{-1,\overline{\gamma}}\right)_{\mathcal{V}}(\Omega)\right)$, $c \in M\left(\tilde{H}^1(\Omega) \to \left(K_2^{-1,\overline{\gamma}}\right)_{\mathcal{V}}(\Omega)\right)$.

Then the solving operator $\mathcal{R}$ *is bounded:*

$$\mathcal{R} : \left(B^{-1,\bar{\epsilon}}_{2,1}\right)_{\mathcal{V}}(\Omega) \to \left(\tilde{N}^{1,\bar{\epsilon}}_{2}\right)_{\mathcal{V}}(\Omega), \quad \bar{\epsilon} = (\epsilon_1, \ldots, \epsilon_d),$$

where

- $\epsilon_i \preceq \sqrt{\varkappa_i}$,
- $\exists c \in (0,1) : \epsilon_i \preceq \gamma_i^c$, $i = \overline{1, \ldots, d}$,
- $\epsilon_j = \epsilon_d \circ \omega_j$, $j = \overline{1, \ldots, d-1}$.

Theorem

Let $\partial\Omega \in C^{0,\overline{\omega}}_{\mathcal{V}}$ *and*

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A3. $\boldsymbol{A} \in C^{0,\overline{\varkappa}}_{\mathcal{V}}(M)$, $\overline{\varkappa} = (\varkappa_1, \dots, \varkappa_d)$, $\varkappa_k$ *be moduli continuity.*

A4. $b \in M\left(L_2(\Omega) \to \left(K^{-1,\overline{\gamma}}_2\right)_{\mathcal{V}}(\Omega)\right)$, $c \in M\left(\tilde{H}^1(\Omega) \to \left(K^{-1,\overline{\gamma}}_2\right)_{\mathcal{V}}(\Omega)\right)$.

Then the solving operator $\mathcal{R}$ *is bounded:*

$$\mathcal{R} : \left(B^{-1,\bar{\epsilon}}_{2,1}\right)_{\mathcal{V}}(\Omega) \to \left(\tilde{N}^{1,\bar{\epsilon}}_2\right)_{\mathcal{V}}(\Omega), \quad \bar{\epsilon} = (\epsilon_1, \dots, \epsilon_d),$$

where

- $\epsilon_i \preceq \sqrt{\varkappa_i}$,
- $\exists c \in (0,1) : \epsilon_i \preceq \gamma_i^c$, $i = \overline{1, \dots, d}$,
- $\epsilon_j = \epsilon_d \circ \omega_j$, $j = \overline{1, \dots, d-1}$.

Theorem

Let $\partial\Omega \in C^{0,\overline{\omega}}_{\mathcal{V}}$ *and*

A1. $\forall x \in M$, $\forall \xi, \eta \in \otimes_{\mathbb{C}} T^*_x M \Rightarrow \boldsymbol{A}(\xi \otimes \bar{\eta}) = \overline{\boldsymbol{A}(\eta \otimes \bar{\xi})}$;

A2. $\exists \alpha > 0 : \forall x \in M \ \forall \xi \in \otimes_{\mathbb{C}} T^*_x M \Rightarrow \alpha \boldsymbol{G}(\xi \otimes \bar{\xi}) \leq \boldsymbol{A}(\xi \otimes \bar{\xi})$;

A3. $\boldsymbol{A} \in C^{0,\overline{\varkappa}}_{\mathcal{V}}(M)$, $\overline{\varkappa} = (\varkappa_1, \ldots, \varkappa_d)$, $\varkappa_k$ *be moduli continuity.*

A4. $b \in M\left(L_2(\Omega) \to \left(K_2^{-1,\overline{\gamma}}\right)_{\mathcal{V}}(\Omega)\right)$, $c \in M\left(\tilde{H}^1(\Omega) \to \left(K_2^{-1,\overline{\gamma}}\right)_{\mathcal{V}}(\Omega)\right)$.

Then the solving operator $\mathcal{R}$ *is bounded:*

$$\mathcal{R} : \left(B^{-1,\overline{\epsilon}}_{2,1}\right)_{\mathcal{V}}(\Omega) \to \left(\tilde{N}^{1,\overline{\epsilon}}_2\right)_{\mathcal{V}}(\Omega), \quad \overline{\epsilon} = (\epsilon_1, \ldots, \epsilon_d),$$

where

- $\epsilon_i \preceq \sqrt{\varkappa_i}$,
- $\exists c \in (0,1) : \epsilon_i \preceq \gamma_i^c$, $i = \overline{1, \ldots, d}$,
- $\epsilon_j = \epsilon_d \circ \omega_j$, $j = \overline{1, \ldots, d-1}$.

Theorem (Holder case)

Let $\partial\Omega \in C^{0,\omega}_{\mathcal{V}}$, $\omega = (\omega_1, \ldots, \omega_d)$, $\omega_k \in (0, 1]$, *and*

A1. $\forall x \in M$, $\forall \xi, \eta \in \otimes_{\mathbb{C}} T^*_x M \Rightarrow \boldsymbol{A}(\xi \otimes \bar{\eta}) = \overline{\boldsymbol{A}(\eta \otimes \bar{\xi})}$;

A2. $\exists \alpha > 0 : \forall x \in M \ \forall \xi \in \otimes_{\mathbb{C}} T^*_x M \Rightarrow \alpha \boldsymbol{G}(\xi \otimes \bar{\xi}) \leq \boldsymbol{A}(\xi \otimes \bar{\xi})$;

A3. $\boldsymbol{A} \in C^{0,\varkappa}_{\mathcal{V}}(M)$, $\varkappa = (\varkappa_1, \ldots, \varkappa_d)$, $\varkappa_k \in (0, 1]$.

A4. $b \in M\left(L_2(\Omega) \to \left(K_2^{-1+\gamma}\right)_{\mathcal{V}}(\Omega)\right)$, $c \in M\left(\tilde{H}^1(\Omega) \to \left(K_2^{-1+\gamma}\right)_{\mathcal{V}}(\Omega)\right)$, $\gamma_k \in (0, 1]$.

Then the solving operator $\mathcal{R}$ *is bounded:*

$$\mathcal{R} : \left(B^{-1+\epsilon}_{2,1}\right)_{\mathcal{V}}(\Omega) \to \left(\tilde{N}^{1+\epsilon}_2\right)_{\mathcal{V}}(\Omega), \quad \epsilon = (\epsilon_1, \ldots, \epsilon_d), \qquad \epsilon_k \in (0, 1),$$

where

- $\epsilon_i \leq \varkappa_i / 2$,
- $\epsilon_i < \gamma_i$, $i = \overline{1, \ldots, d}$,
- $\epsilon_j = \epsilon_d \cdot \omega_j$, $j = \overline{1, \ldots, d-1}$.

Due to M. L. Goldman results, it is possible to get sufficient conditions for satisfying **(A4)**.

For example, $L_q(\Omega) \hookrightarrow M(L_2(\Omega) \to K_2^{-1,\overline{\gamma}}(\Omega))$, if

$$\left\| \frac{\inf_{t_1 \cdots t_d = t} \max_{1 \le j \le d} \frac{t_j}{\gamma_j(t_j)}}{\sqrt{t}} \right\|_{L_r[0,1]} < \infty, \quad 1/q + 1/r = 1/2.$$

Similarly, one has $K_{s_1}^{-1,\overline{\gamma}}(\Omega) \hookrightarrow M(\tilde{H}^1(\Omega) \to K_2^{-1,\overline{\gamma}}(\Omega))$, if for $1/s_1 + 1/s_2 + 1/s_3 = 1$ the following estimate holds

$$\left\| \frac{\inf_{t_1 \cdots t_d = t} \max_{1 \le j \le d} \frac{t_j}{\gamma_j(t_j)}}{\sqrt{t}} \right\|_{L_{s_2}[0,1]} + \left\| \frac{\inf_{t_1 \cdots t_d = t} \max_{1 \le j \le d} \gamma_j(t_j)}{\sqrt{t}} \right\|_{L_{s_3}[0,1]} < \infty,$$

If $\gamma_i(\cdot) = (\cdot)^{\beta_i}$, then for some $\varepsilon > 0$

$$L_{\sum \frac{1}{\beta_i} + \varepsilon}(\Omega) \hookrightarrow M(L_2(\Omega) \to K_2^{-1,\overline{\gamma}}(\Omega))$$
$$K_{d+\varepsilon}^{-1,\overline{\gamma}}(\Omega) \hookrightarrow M(\tilde{H}^1(\Omega) \to K_2^{-1,\overline{\gamma}}(\Omega)).$$

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