Some properties on multi-radial functions

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Strauss' Radial Lemma

Strauss' Radial Lemma (1977): Let $N > 2$. Every radial function $f \in W_2^1(\mathbb{R}^N)$ is almost everywhere equal to a function $\widetilde{f}$, continuous for $x \neq 0$, such that

$$|\widetilde{f}(x)| \le c\,|x|^{1-\frac{N}{2}}\,\|\,\nabla f\,|L_2(\mathbb{R}^N)\|\,,$$

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(a) the existence of a representative of f, which is continuous outside the origin;

(b) the decay of f near infinity;

(c) the limited unboundedness near the origin.

(d) points (a)-(c) implies compactness of some of Sobolev embeddings of "radial parts" of inhomogeneous Sobolev spaces.

Radial lemma - the simple example $p = 1$

Let $f = g(r(x)) \in C_0^\infty(\mathbb{R}^N)$ be smooth radial function. Then

$$\frac{\partial f}{\partial x_i}(x) = g'(r)\,\frac{x_i}{r}, \quad r = |x| > 0\,, \qquad i = 1, \dots, d\,.$$

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This extends to the closure of radial $C_0^\infty(\mathbb{R}^d)$ in the gradient norm:

$$|x|^{N-1}\,|f(x)| = r^{N-1}\,|g(r)| \le c_N \int_{|x|>r} |\nabla f(x)|\,dx \le c_N\,\|\,\nabla f(x)\,\|_1\,.$$

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(i) *Then there exists a constant* $c > 0$ *s.t.*

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holds for all radial $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^N)$, *for all* $x \neq 0$.

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holds for all radial $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^N)$, for all $x \neq 0$.
(ii) *There exist a positive constant $c > 0$ and a function $f \in R\dot{B}^s_{p,\infty}(\mathbb{R}^N)$, s.t.*

$$|f(x)| \geq c\,|x|^{s-\frac{N}{p}}\,\|\,f\,|\dot{B}^s_{p,\infty}(\mathbb{R}^N)\|$$

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Then the following assertions are equivalent.
(i) *There exists a constant c such that*

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Remark For $p = 2$ cf. Y.Cho, T.Ozawa (2009).

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\begin{aligned}
G = SO(\gamma) = SO(\mathbb{R}^{\gamma_1}) \times \ldots \times SO(\mathbb{R}^{\gamma_m}), \\
\gamma = (\gamma_1, \ldots, \gamma_m), \qquad N = |\gamma| = \gamma_1 + \ldots + \gamma_m \\
g(x) = (g_1(x_1), \ldots, g_m(x_m)), \quad g = (g_1, \ldots, g_m) \in G, \\
x = (x_1, \ldots, x_m) \in \mathbb{R}^N = \mathbb{R}^{\gamma_1} \times \cdots \times \mathbb{R}^{\gamma_m},
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- If E denotes a space of distributions on $\mathbb{R}^N$ then by E_γ we mean the subspace of $SO(\gamma)$-invariant distributions in E and we endow this subspace with the same norm as the original space.

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- The orbits of $G = SO(\gamma)$:
  $G \cdot x = x \Leftrightarrow x = 0$; $\dim G \cdot 0 = \{0\}$;
  if $r_i(x) \neq 0$ for any i then $\dim G \cdot x = \Pi_{i=1}^m (\gamma_i - 1)$
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- We put also for $1 \leq n \leq m$

$$R_n(x) = \prod_{i=1}^{n} r_i(x)^{\gamma_i - 1}, \quad \text{and} \qquad r_{min}(x) \ = \ \min\{r_i(x) : 1 \leq i \leq m\}.$$

More general symmetries - Sobolev spaces

- The spaces $\dot{H}^{s,p}_{\gamma}(\mathbb{R}^N)$, $s > 0$, $p > 1$ are defined as the completion of $C^\infty_{0,\gamma}(\mathbb{R}^N)$ in the norm

$$\|u\|_{s,p} = \|(-\Delta)^s u\|_p = \|F^{-1}(|\xi|^s Fu)\|_p,$$

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- The space $\dot{H}^{s,p}_{\gamma}(\mathbb{R}^N)$ can be identified as subspace of homogeneous Sobolev space $\dot{H}^{s,p}(\mathbb{R}^N)$ defined as the completion of $C^{\infty}_{0}(\mathbb{R}^N)$. The space $\dot{H}^{s,p}(\mathbb{R}^N)$ is a spaces of functions if $sp < N$.

Multiradial functions -general remarks

Let $f \in \dot{H}^{s,p}_{\gamma}(\mathbb{R}^N)$, $p > 1$, $m < sp < N$. If $R_m(x)$ is bounded away from zero, then f is locally a $H^{s,p}$-function of m variables $r_1, \dots, r_m$ and is therefore continuous in such region since $m < sp$.

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Assume that $r_1(x) \geq \dots \geq r_m(x)$ and consider a region where $R_n(x)$, $1 \leq n < m$, is bounded away from zero. Let $d_n := \sum_{i=n+1}^{m} \gamma_i + n$. In such region, f can be considered locally as a $H^{s,p}$-function of d_n variables $x_1, \dots, x_{\sum_{i=n+1}^{m} \gamma_i}, r_1, \dots, r_n$, and therefore f is continuous whenever $R_n(x) \neq 0$ if $d_n < sp$.

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Since d_n is a monotone decreasing function of n, $d_m = m$, and $d_0 = N$, there exists $n^* \in \{1, \ldots, m-1\}$, which is the smallest n such that $d_n \leq sp$, such that f is continuous whenever $R_{n^*-1}(x) \neq 0$, but may be discontinuous at the orbits
$\Gamma = \{x : \ r_{n^*}(x) = \cdots = r_m(x) = 0, r_i(x) = \rho_i > 0, i = 1, \ldots, n^* - 1\}$.

Strauss lemma for block-radial functions

Theorem

Let $s > 0$, $m \in \mathbb{N}$, $p > 1$, $m < sp < N$ *and assume that* $\gamma_i \geq 2$, $i = 1, \ldots, m$. *Assume also that* $sp \neq d_J$ *for any* $J \subset \{1, \ldots, m\}$.

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$$|f(x)| \leq C \left(|x|^{s-N/p} + R_m(x)^{-1/p} r_{\min}(x)^{s-m/p} \right) \|f\|_{s,p}, \tag{2}$$

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The radial case corresponds to $m = 1$. Then $R_m(x) = |x|^{N-1}$ and $r_{min}(x) = |x|$ so we got the Strauss estimates for radial functions.

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If $r_1(x) = \ldots = r_m(x)$. Then $r_m(x) = \frac{|x|}{\sqrt{m}}$ and $R_m(x)^{-1/p} r_{\min}(x)^{s-m/p} = c|x|^{s-N/p}$. So in that case we have the Strauss estimate.

Strauss lemma for block-radial functions II

Corollary

Assume the conditions of Theorem 1. If, additionally, $sp \geq N - \gamma_i + 1$ for all $i = 1, \dots, m$, then inequality (2) becomes

$$|f(x)| \leq C|x|^{s-N/p}\|f\|_{s,p}\,. \tag{3}$$

If, however, $sp \leq m + \gamma_i - 1$ for all $i = 1, \dots, m$, then inequality (2) becomes

$$|f(x)| \leq C\, R_m(x)^{-1/p} r_{\min}(x)^{s-m/p}\|f\|_{s,p}\,. \tag{4}$$

Strauss inequality - bi-radial case

Corollary

Assume that $2 = m < sp < N$ *and that* $\gamma_i \geq 2$, $i = 1, 2$. *If* $sp > \max\{\gamma_1, \gamma_2\} + 1$, *then inequality* (2) *becomes*

$$|f(x)| \leq C\,|x|^{s-N/p}\|f\|_{s,p}\,. \tag{5}$$

If $sp < \min\{\gamma_1, \gamma_2\} + 1$, *then inequality* (2) *becomes*

$$|f(x)| \leq C\,R_2(x)^{-1/p} r_{\min}(x)^{s-2/p}\|f\|_{s,p}. \tag{6}$$

Strauss inequality - limiting case $sp = d_J$

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There exists $C > 0$, $C = C(\gamma, s, p)$ *such that for every* $f \in \dot{H}^{s,p}_{\gamma}(\mathbb{R}^N)$, *and every* $x \in U_J$

$$|f(x)| \leq \left(1 + \log \frac{\min_{i \notin J} r_i(x)}{\max_{i \in J} r_i(x)}\right) R_J(x)^{-1/p} \|f\|_{s,p}. \tag{7}$$

Bi-radial case

Corollary

Let $s > 0$, $m = 2 < sp < N$ *and assume that* $\gamma_i \geq 2$, $i = 1, 2$. *Let* $d_* = \min\{\gamma_1, \gamma_2\} + 1$ *and* $d^* = \max\{\gamma_1, \gamma_2\} + 1$. *Let* $d_* < sp < d^*$ *then there exist* $C > 0$ *such that for any* $f \in \dot{H}^{s,p}(\mathbb{R}^N)$,

$$|f(x)| \leq \begin{cases} |x|^{s-\frac{N}{p}} \|f\|_{s,p}, & \text{if } r_{\min}(x) = r_i, \text{ and } \gamma_i = \min(\gamma_1, \gamma_2), \\ |x|^{\frac{d^*-N}{p}} r_{\min}(x)^{\frac{sp-d^*}{p}} \|f\|_{s,p}, & \text{if } r_{\min}(x) = r_i, \text{ and } \gamma_i = \max(\gamma_1, \gamma_2). \end{cases}$$

Let $sp = d_* < d^*$ *or* $sp = d^* > d_*$, *then*

$$|f(x)| \leq \begin{cases} \left(1 + \left|\log \frac{r_1(x)}{r_2(x)}\right|\right) |x|^{s-\frac{N}{p}} \|f\|_{s,p}, & x \in \mathbb{R}^N : \; r_i(x) \leq \frac{1}{\sqrt{2}}|x| \\ |x|^{s-\frac{N}{p}} \|f\|_{s,p}, & x \in \mathbb{R}^N : \; r_i(x) \geq \frac{1}{\sqrt{2}}|x|, \end{cases}$$

if $\gamma_i = d_* - 1$ *or* $\gamma_i = d^* - 1$ *respectively.*

Bi-radial case

Corollary

Let $s > 0$, $m = 2 < sp < N$ *and assume that* $\gamma_i \geq 2$, $i = 1, 2$. *Let* $d_* = \min\{\gamma_1, \gamma_2\} + 1$ *and* $d^* = \max\{\gamma_1, \gamma_2\} + 1$.
Let $sp = d^* = d_*$. *Then*

$$|f(x)| \leq \left(1 + \left|\log \frac{r_1(x)}{r_2(x)}\right|\right) |x|^{s-\frac{N}{p}} \|f\|_{s,p}, \quad x \in \mathbb{R}^N \; r_1(x) \cdot r_2(x) > 0$$

Strauss inequality - optimality of the estimates

- For any $x \neq 0$ there exists a smooth compactly supported radial function ψ such that $\psi(x) = 1$ and

$$\|\psi\|_{s,p} \sim |x|^{\frac{N}{p}-s}$$

with constants independent of ϕ and x.

- Let $sp \neq d_J$ for any $J \subset 1, \ldots, m$. For any x such that $R_m(x) \neq 0$ there exists a smooth compactly supported $SO(\gamma)$-invariant function ψ such that $\psi(x) = 1$ and

$$\|\psi\|_{s,p} \sim R_m(x)^{1/p} r_{min}(x)^{\frac{m}{p}-s}$$

with constants independent of ψ and x.

Sobolev spaces once more

- $\dot{H}^{1,p}_{0,\gamma}(\mathbb{R}^N)$ denotes the completion of $C^{\infty}_{0,\gamma}(\mathbb{R}^N \setminus Y(\gamma))$ in the gradient norm $\|\nabla f\|_p$, where

$$Y(\gamma) = \bigcup_{k:\gamma_k \geq 2} Y_k \subset \mathbb{R}^N.$$

Y_k is a hyperplane of codimension γ_k defined by $r_k = 0$.

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  Y_k is a hyperplane of codimension γ_k defined by $r_k = 0$.
- If $1 < p < \min\{\gamma_k : \gamma_k \geq 2, k = 1, \ldots, m\}$ and $1 < m < N$, then $\dot{H}^{1,p}_{0,\gamma}(\mathbb{R}^N) = \dot{H}^{1,p}_{\gamma}(\mathbb{R}^N)$.

Block radial symmetry- Hardy's inequalities

Theorem (L.S., C.Tintarev (2016))

Let $1 < m < N$ *and* $1 \leq p < \infty$. *There exist* $C > 0$, $C = C(\gamma)$ *if* $2 \leq p < \infty$, *such that for all* $f \in C^\infty_{0,\gamma}(\mathbb{R}^N \setminus Y(\gamma))$,

$$\Big(\int_{\mathbb{R}^N} \frac{|f(x)|^p}{r_\gamma(x)^p} \Big)^{1/p} \leq C \Big(\int_{\mathbb{R}^N} |\nabla f(x)|^p dx \Big)^{1/p} \qquad (8)$$

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Theorem (L.S., C.Tintarev (2016))

Let $1 < m < N$ *and* $1 < p < \min\{\gamma_k : \gamma_k \ge 2\}$. *Then there exists a positive constant* C *such that for each* $f \in \dot{H}^{1,p}_{\gamma}(\mathbb{R}^N)$,

$$\Big(\int_{\mathbb{R}^N} \frac{|f(x)|^p}{r_\gamma(x)^p} dx \Big)^{1/p} \le C \Big(\int_{\mathbb{R}^N} |\nabla u(x)|^p dx \Big)^{1/p}. \qquad (9)$$

Moreover $C = C(\gamma)$ *if* $2 \le p < \infty$. *Here* $r_\gamma(x) = R_m(x)^{1/(N-m)}$.

Block radial symmetry - CKN inequalities

Theorem (L.S., C.Tintarev (2016))

Let $1 < m < N$, $1 \le p < \infty$, $p \le q < \infty$, *and let* $q \le p_m^* := \frac{pm}{m-p}$ *whenever* $p < m$. *Then there exists a constant* $C > 0$, *uniform with respect to* $p > 2$, *such that for every* $f \in \dot{H}^{1,p}_{0,\gamma}(\mathbb{R}^N)$,

$$\left(\int_{\mathbb{R}^N} \left(\frac{|u(x)|}{r_\gamma(x)^{|\gamma|(\frac{1}{q}-\frac{1}{p})+1}} \right)^q dx \right)^{1/q} \le C \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p dx \right)^{1/p}. \qquad (10)$$

If $p \neq m$ *then the constant* $C > 0$ *is independent of* p.
Moreover if $p < \min\{\gamma_k : \gamma_k \ge 2\}$ *then the inequality* (10) *holds for any* $f \in \dot{H}^1_{p,\gamma}(\mathbb{R}^N)$.

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