Interpolation of uniformly absolutely continuous operators

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Motivation

$\Omega \subset \mathbb{R}^n, \qquad \Omega \in C^{0,1}, \qquad 1 \le p < n,$

$$W^{1,p}(\Omega) \hookrightarrow L^{\frac{np}{n-p},p}(\Omega)$$
$$L^p(\Omega) \hookrightarrow L^p(\Omega)$$

$1 \le q < \frac{np}{n-p}$

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$$
$$L^p(\Omega) \hookrightarrow L^p(\Omega)$$

$1 \le q < p$

$$W^{1,p}(\Omega) \hookrightarrow L^{\frac{np}{n-p},p}(\Omega)$$
$$L^p(\Omega) \overset{*}{\hookrightarrow} L^q(\Omega)$$

sequentially compactness in $BFS_a \Leftrightarrow$

$\Leftrightarrow$ (sequentially compactness in measure + UAC)

Banach function space

Definition

Let $(\mathcal{R}, \mu)$ be a measure space. We denote by $\mathcal{M}(\mathcal{R})$ the set of all measurable functions on $\mathcal{R}$.

A Banach space $X(\mathcal{R}) \subset \mathcal{M}(\mathcal{R})$, equipped with the norm $\|\cdot\|_{X(\mathcal{R})}$, is said to be a Banach function space over the measure space $(\mathcal{R}, \mu)$ if the following axioms hold for every $f, g, f_n \in \mathcal{M}(\mathcal{R})$:

$$0 \leq g \leq f \; \mu\text{-a.e. implies } \|g\|_{X(\mathcal{R})} \leq \|f\|_{X(\mathcal{R})}; \tag{P1}$$

$$0 \leq f_n \nearrow f \; \mu\text{-a.e. implies } \|f_n\|_{X(\mathcal{R})} \nearrow \|f\|_{X(\mathcal{R})}; \tag{P2}$$

$$\|\chi_E\|_{X(\mathcal{R})} < \infty \text{ for every } E \subset \mathcal{R}, \; \mu(E) < \infty; \tag{P3}$$

$$\text{for every } E \subset \mathcal{R}, \text{with } \mu(E) < \infty, \exists C_E \in (0, \infty) \text{ such that} \tag{P4}$$

$$\int_E f \leq C_E \|f\|_{X(\mathcal{R})} \quad \text{for every } f \in X(\mathcal{R}).$$

Banach lattice

Definition

If a Banach space $X(\mathcal{R}) \subset \mathcal{M}(\mathcal{R})$ satisfies (P1), then we say that X is a Banach lattice over the measure space $(\mathcal{R}, \mu)$.

Moreover, if a Banach lattice X also satisfies (P2), we say that it has the Fatou property.

Uniformly absolutely continuous set

Definition

A function f in a Banach function space $X(\mathcal{R})$ is said to have absolutely continuous norm in $X(\mathcal{R})$ if $\|f\chi_{E_n}\|_{X(\mathcal{R})} \to 0$ for every sequence $\{E_n\}$ of subsets of $\mathcal{R}$ such that $\chi_{E_n} \to 0$ μ–a.e.

A set $M \subset X(\mathcal{R})$ is uniformly absolutely continuous in $X(\mathcal{R})$ if

$\lim_{n\to\infty} \sup_{f\in M} \|f\chi_{E_n}\|_{X(\mathcal{R})} = 0$ for every sequence E_n such that $\chi_{E_n} \to 0$ μ-a.e.

Uniformly absolutely continuous operator

Definition

Let A be a Banach space and let B be a Banach function space over a measure space $(\mathcal{R}, \mu)$. Let T be a bounded linear operator defined on A with values in B (notation $T : A \to B$). Then T is said to be uniformly absolutely continuous (notation $T : A \xrightarrow{*} B$) if the image under T of the unit ball in A is uniformly absolutely continuous in B. This means that

$$\lim_{n\to\infty} \sup_{\|f\|_A \le 1} \|\chi_{E_n} \cdot Tf\|_B = 0$$

for every sequence $\{E_n\}$ of subsets of $\mathcal{R}$ such that $\chi_{E_n} \to 0$ μ–a.e.

The class Φ

Definition

We say that a function $\varphi : [0,\infty) \times [0,\infty) \to [0,\infty)$, $\varphi \not\equiv 0$, belongs to the class Φ if it has the following properties:

(i) $\varphi(0,0) = 0$,

(ii) $\varphi(s,t)$ is almost non-decreasing in each variable,

(iii) $\varphi(s,t)$ is positively homogeneous of degree 1, that is,

$$\varphi(\lambda s, \lambda t) = \lambda \varphi(s,t) \quad \text{for every} \quad \lambda, s, t \in [0,\infty).$$

The cone Q

Definition

A non-negative function h on $[0,\infty)$ is called quasiconcave (notation $h \in Q$) if $h(t) = 0$ if and only if $t = 0$, h is non-decreasing on $(0,\infty)$ and $\frac{h(t)}{t}$ is non-increasing on $(0,\infty)$.

Remarks

(i) If $\varphi \in \Phi$, then φ is quasiconcave in each variable.
Moreover, $\varphi(s,t) > 0$ if $s \neq 0$ and $t \neq 0$ and the function

$$\varphi^*(t) := \varphi(1,t) \quad \text{for all} \quad t \in [0,\infty)$$

is quasiconcave and

$$\varphi(s,t) = s\varphi\left(1, \frac{t}{s}\right) = s\varphi^*\left(\frac{t}{s}\right) \quad \text{for all } s \in (0,\infty).$$

(ii) On the other hand, if $\varphi^* : [0,\infty) \to [0,\infty)$ is quasiconcave, then

$$\varphi(s,t) := \begin{cases} s\varphi^*\left(\frac{t}{s}\right) & \text{if } s,t \in (0,\infty), \\ 0 & \text{if } s = 0 \text{ or } t = 0, \end{cases}$$

satisfies $\varphi \in \Phi$.

(iii) If $\varphi \in \Phi$, then there exists a constant $C \in (0,\infty)$ such that

$$\varphi(s,t) \leq C \max\{s,t\} \quad \text{for every} \quad s,t \in [0,\infty).$$

Type of interpolation method

Definition

Let $\varphi \in \Phi$, let $\mathcal{C}$ be a category of Banach spaces and let $\mathcal{C}_1$ be the subcategory of compatible pairs of spaces from $\mathcal{C}$.
An interpolation method $\mathcal{F}$ is said to be of type φ on $\mathcal{C}_1$ if, for all $s, t \in [0, \infty)$,

$$\sup\{\|T\|_{\mathcal{F}(\bar{A}) \to \mathcal{F}(\bar{B})};\ \|T\|_{A_0 \to B_0} \le s,\ \|T\|_{A_1 \to B_1} \le t, \bar{A}, \bar{B} \in \mathcal{C}_1,\ T \in \mathcal{L}(\bar{A}, \bar{B})\} \lesssim \varphi(s, t)$$

(note that $\bar{A} = (A_0, A_1),\ \bar{B} = (B_0, B_1)$).
An interpolation method $\mathcal{F}$ is said to be of sharp type φ on $\mathcal{C}_1$ if, for all $s, t \in [0, \infty)$,

$$\sup\{\|T\|_{\mathcal{F}(\bar{A}) \to \mathcal{F}(\bar{B})};\ \|T\|_{A_0 \to B_0} \le s,\ \|T\|_{A_1 \to B_1} \le t, \bar{A}, \bar{B} \in \mathcal{C}_1,\ T \in \mathcal{L}(\bar{A}, \bar{B})\} \approx \varphi(s, t).$$

First results

Theorem 1

Let $\varphi \in \Phi$, let $\mathcal{C}$ be a category of Banach spaces and let $\mathcal{C}_1$ be the subcategory of compatible pairs of spaces from $\mathcal{C}$. Let $\bar{A} = (A_0, A_1) \in \mathcal{C}_1$ and $\bar{B} = (B_0, B_1) \in \mathcal{C}_1$. Suppose that B_0 and B_1 are Banach function spaces over the same measure space $(\mathcal{R}, \mu)$. Assume that $\mathcal{F}$ is an interpolation method of type φ on $\mathcal{C}_1$, where φ satisfies

$$\lim_{s \to 0+} \varphi(s, t) = 0 \qquad \text{for any fixed } t \in (0, \infty).$$

If T is a linear operator such that

$$T : A_0 \stackrel{*}{\to} B_0$$

and

$$T : A_1 \to B_1,$$

then

$$T : \mathcal{F}(\bar{A}) \stackrel{*}{\to} \mathcal{F}(\bar{B}).$$

Theorem 2

Let $\varphi \in \Phi$, let $\mathcal{C}$ be a category of Banach spaces and let $\mathcal{C}_1$ be the subcategory of compatible pairs of spaces from $\mathcal{C}$. Let $\bar{A} = (A_0, A_1) \in \mathcal{C}_1$ and $\bar{B} = (B_0, B_1) \in \mathcal{C}_1$. Suppose that B_0 and B_1 are Banach function spaces over the same measure space $(\mathcal{R}, \mu)$. Assume that $\mathcal{F}$ is an interpolation method of type φ on $\mathcal{C}_1$, where φ satisfies

$$\lim_{t \to 0+} \varphi(s, t) = 0 \qquad \text{for any fixed } s \in (0, \infty).$$

If T is a linear operator such that

$$T : A_0 \to B_0$$

and

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then

$$T : \mathcal{F}(\bar{A}) \overset{*}{\to} \mathcal{F}(\bar{B}).$$

Besov space $B^{0,\beta}_{p,q}(\mathbb{R}^n)$

Definition

Let $p \in [1, \infty)$ and $q \in [1, \infty]$. Denote $\ell(t) := (1 + |\log t|)$, $t \in (0, 1)$, and let $\beta \in \mathbb{R}$ be such that

$$\beta + \frac{1}{q} \geq 0 \quad \text{if } q < \infty \quad \text{and} \quad \beta \in (0, \infty) \quad \text{if } q = \infty.$$

Let $\omega_1(f, t)_p$ be the value of the first-order modulus of continuity of a function f at t with respect to $L^p(\mathbb{R}^n)$, defined by

$$\omega_1(f, t)_p := \sup_{|h| \leq t} \|f(x + h) - f(x)\|_{p,\mathbb{R}^n},$$

where $h \in \mathbb{R}^n$ and $t \in ([0, \infty)$.

The Besov space

$$B^{0,\beta}_{p,q}(\mathbb{R}^n) := \{f \in L^p(\mathbb{R}^n) : \ \|f\|_{B^{0,\beta}_{p,q}} < \infty\},$$

where

$$\|f\|_{B^{0,\beta}_{p,q}} := \|f\|_{p,\mathbb{R}^n} + \|t^{-\frac{1}{q}} \ell^{\beta}(t) \omega_1(f, t)_p\|_{q,(0,1)}.$$

Application

Theorem 3

Let $p \in [1,\infty)$, $q \in [1,\infty]$ *and let* $\beta \in \mathbb{R}$ *be such that*

$$\beta + \frac{1}{q} \geq 0 \quad \text{if } q < \infty \quad \text{and} \quad \beta > 0 \quad \text{if } q = \infty.$$

If $\Omega \subset \mathbb{R}^n$ *is a bounded domain and* $\delta < 0$, *then*

$$B^{0,\beta}_{p,q}(\mathbb{R}^n) \hookrightarrow\hookrightarrow Y(\Omega),$$

where

$$Y(\Omega) := \left\{ f \in L^p(logL)^\delta(\Omega); \|f\|_Y < \infty \right\}$$

and

$$\|f\|_Y := \|t^{-\frac{1}{q}}\ell^\beta(t)\|\ell^\delta(\tau)f^*(\tau)\|_{p,(0,t)}\|_{q,(0,1)}.$$

In particular,

$$B^{0,\beta}_{p,q}(\mathbb{R}^n) \hookrightarrow\hookrightarrow L^{p,q}(logL)^{\beta+\delta+\frac{1}{\max\{p,q\}}}(\Omega).$$

The abstract K-method

Definition

Let Z be a Banach lattice over the measure space $((0,\infty), dy)$ such that

$$0 < \|\min\{1, y\}\|_Z < \infty.$$

Let $\bar{A} = (A_0, A_1)$ be a compatible pair of Banach spaces.

The abstract K-method of interpolation $(\cdot,\cdot)_{Z;K}$ is defined by

$$\bar{A}_{Z,K} = (A_0, A_1)_{Z,K} := \{a \in A_0 + A_1 : \ \|a\|_{(A_0,A_1)_{Z;K}} < \infty\},$$

where

$$\|a\|_{Z;K} = \|a\|_{(A_0,A_1)_{Z;K}} := \left\|K(a, y; \bar{A})\right\|_{Z((0,\infty);dy)}$$

and $K(a, \cdot\,; \bar{A})$ stands for the K-functional.

$\bar{A}_{Z,K}$ is the interpolation space with respect to the pair $\bar{A}$.

$\tilde{Q}$-abundant Banach pair $\bar{A}$

Q .... the cone of quasiconcave functions on $[0, \infty)$

Definition

Let $\tilde{Q}$ be a subcone of Q. We say that a Banach pair $\bar{A} = (A_0, A_1)$ is $\tilde{Q}$-abundant if there exists a constant $C \in (0, \infty)$ such that for every function $h \in \tilde{Q}$ there exists some $a \in A_0 + A_1$ so that

$$C^{-1} h(t) \leq K(a, t; \bar{A}) \leq C\, h(t) \quad \text{for all } t \in (0, \infty).$$

Let Q_1 and Q_2 be subcones of Q such that $Q_2 \subset Q_1$. Then

$$\bar{A} \text{ is } Q_1\text{-abundant} \quad \Rightarrow \quad \bar{A} \text{ is also } Q_2\text{-abundant.}$$

Important subcones:

$$Q_0 := \{h \in Q : \lim_{y \to 0+} h(y) = 0\}$$

$$Q_{0,\infty} := \{h \in Q_0 : \lim_{y \to \infty} y^{-1} h(y) = 0\}$$

Lemma

Consider the following compatible pairs of function spaces:

- $\bar{A} = (A_0, A_1) := (L^1(0,\infty), L^\infty(0,\infty))$;
- $\bar{B} = (B_0, B_1) := (L^1(0,\infty), L^1((0,\infty); \frac{1}{y}, dy))$;
- $\bar{C} = (C_0, C_1) := (L^\infty(0,\infty), L^\infty((0,\infty); \frac{1}{y}, dy))$.

Then $\bar{A}$ is Q_0-abundant, $\bar{B}$ is $Q_{0,\infty}$-abundant and $\bar{C}$ is Q_0-abundant.

Remark

$$K(f,t;A_0,A_1) = \int_0^t f^*(y)\,dy$$

$$K(f,t;B_0,B_1) \approx \int_0^\infty |f(y)| \min\left\{1, \frac{t}{y}\right\} dy = \int_0^t \left(\int_y^\infty \frac{|f(s)|}{s}\,ds \right) dy$$

$$K(f,t;C_0,C_1) \approx \sup_{y\in(0,\infty)} |f(y)| \min\left\{1, \frac{t}{y}\right\} = \sup_{y\in(0,t]} y \sup_{s\in[y,\infty)} \frac{|f(s)|}{s}$$

Dilation operator E_t

Definition

Given a Banach lattice Z over the measure space $((0,\infty), dy)$ and $t \in (0,\infty)$, we denote by E_t the dilation operator on Z defined by

$$(E_t f)(y) = f(ty), \quad y \in (0,\infty).$$

If $\tilde{Q}$ is a subcone of Q, $Z_{\tilde{Q}} := Z \cap \tilde{Q}$ and $T : Z \to Z$ is an operator, then we put

$$\|T\|_{Z_{\tilde{Q}}} := \sup_{h \in Z \cap \tilde{Q}} \frac{\|Th\|_Z}{\|h\|_Z}.$$

Type of the abstract K-method

Theorem 4

Let Z be a Banach lattice over the measure space $((0,\infty), dy)$ satisfying

$$0 < \| \min\{1, y\}\|_Z < \infty.$$

Let $\mathcal{C}$ be a category of Banach spaces and let $\mathcal{C}_1$ be the subcategory of compatible pairs of spaces from $\mathcal{C}$. Then the abstract K-method is of type φ on $\mathcal{C}_1$, where

$$\varphi(s,t) := \begin{cases} s\|E_{\frac{t}{s}}\|_{Z_Q} & \text{if } s,t \in (0,\infty), \\ 0 & \text{if } s = 0 \text{ or } t = 0. \end{cases}$$

Moreover, if Z has the Fatou property and $\mathcal{C}$ is such that $\mathcal{C}_1$ contains at least one of the pairs $\bar{A}, \bar{B}, \bar{C}$ from Lemma*, then the abstract K-method is of sharp type φ.*

Weighted Lebesgue space

Definition

$$\mathcal{W}(0,\infty) := \{f \in \mathcal{M}(0,\infty) : \ 0 < f < \infty \text{ a.e. on } (0,\infty)\}$$

If $v \in \mathcal{W}(0,\infty)$ and $q \in [1,\infty]$, then

$$L^q((0,\infty); v(y), dy) := \{f \in \mathcal{M}(0,\infty) : \ \|f\|_{q,v,(0,\infty)} < \infty\},$$

where

$$\|f\|_{q,v,(0,\infty)} := \|f(y)v(y)\|_{q,(0,\infty)}.$$

The real interpolation method with a function parameter

A particular case of the abstract K-method with

$$Z = L^q\left((0,\infty); \tfrac{1}{w(y)y^{\frac{1}{q}}}, dy\right), \quad \text{where} \quad w \in \mathcal{W}(0,\infty),\ q \in [1,\infty].$$

Thus, the weight w and q are such that

$$0 < \left\| \frac{\min\{1, y\}}{w(y)y^{\frac{1}{q}}} \right\|_{q,(0,\infty)} < \infty.$$

Gustavsson (1978):
The weight w was assumed to be continuous, non-decreasing, and to satisfy the condition

$$\int_0^\infty \overline{w}(t) \min\left\{1, \frac{1}{t}\right\} \frac{dt}{t} < \infty,$$

where $\quad \overline{w}(t) := \sup_{s\in(0,\infty)} \frac{w(st)}{w(s)}$.

The norm of dilation operator - the particular case

Theorem 5

Let $Z = L^q\left((0,\infty); \frac{1}{w(y)y^{\frac{1}{q}}}, dy\right)$, *where* $w \in \mathcal{W}(0,\infty)$, $q \in [1,\infty]$ *and*

$$0 < \left\| \frac{\min\{1, y\}}{w(y)y^{\frac{1}{q}}} \right\|_{q,(0,\infty)} < \infty.$$

Then

$$\|E_\tau\|_{Z_{\tilde{Q}}} \approx \sup_{\alpha \in (0,\infty)} \frac{\|\min\{1, \alpha\tau \cdot\}\|_Z}{\|\min\{1, \alpha \cdot\}\|_Z}$$

for all $\tau \in (0,\infty)$ *and any* $\tilde{Q} \in \{Q_{0,\infty}, Q_0, Q\}$.

Corollary

Let $w \in \mathcal{W}(0,\infty)$ and $q \in [1,\infty]$ be such that

$$0 < \left\| \frac{\min\{1,y\}}{w(y)y^{\frac{1}{q}}} \right\|_{q,(0,\infty)} < \infty$$

holds. Let $\mathcal{C}$ be a category of Banach spaces such that $\mathcal{C}_1$ contains at least one of the pairs $\bar{A}$, $\bar{B}$, $\bar{C}$ from Lemma. Then the real interpolation method with the function parameter w is of sharp type φ on $\mathcal{C}_1$, where

$$\varphi(s,t) = \begin{cases} sg\left(\frac{t}{s}\right) & \text{if } s,t \in (0,\infty), \\ 0 & \text{if } s=0 \text{ or } t=0, \end{cases}$$

and

$$g(\tau) := \sup_{\alpha \in (0,\infty)} \frac{\left\| \frac{\min\{1,\alpha\tau\,\cdot\}}{w(y)y^{\frac{1}{q}}} \right\|_{q,(0,\infty)}}{\left\| \frac{\min\{1,\alpha\,\cdot\}}{w(y)y^{\frac{1}{q}}} \right\|_{q,(0,\infty)}} \quad \text{for all } \tau \in (0,\infty).$$

Example

Let
$$\beta_0 + \frac{1}{q} \geq 0 \quad \text{if} \quad q < \infty \qquad \text{or} \qquad \beta_0 > 0 \quad \text{if} \quad q = \infty,$$
$$\beta_\infty + \frac{1}{q} < 0 \quad \text{if} \quad q < \infty \qquad \text{or} \qquad \beta_\infty \leq 0 \quad \text{if} \quad q = \infty.$$

Let
$$w(t) = \ell^{-(\beta_0,\beta_\infty)}(t) \quad \text{for} \quad t \in (0,\infty).$$

Hence,
$$Z = L^q((0,\infty); t^{-\frac{1}{q}} \ell^{(\beta_0,\beta_\infty)}(t), dt) \quad \text{and} \quad 0 < \left\| \frac{\min\{1,y\}}{w(y) y^{\frac{1}{q}}} \right\|_{q,(0,\infty)} < \infty.$$

Then the real interpolation method with the function parameter w is of sharp type φ, where

$$\varphi(s,t) = s \qquad \text{for all } s,t \in (0,\infty) \text{ with } \frac{t}{s} \in (0,1],$$
$$\varphi(s,t) = s \ell^{\beta_0 - \beta_\infty}\left(\frac{t}{s}\right) \qquad \text{for all } s,t \in (0,\infty) \text{ with } \frac{t}{s} \in (1,\infty).$$

Thus,
$$\lim_{s \to 0+} \varphi(s,t) = 0 \qquad \text{for any fixed } t \in (0,\infty).$$

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