Pointwise Multipliers for Sobolev and Besov Spaces of Dominating Mixed Smoothness

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Overview

In my talk

- we study the algebra property of Sobolev spaces S^m_pW and Besov spaces $S^t_{p,p}B$ of dominating mixed smoothness
- and we shall give the characterization for the space of all pointwise multipliers of these spaces.

The spaces S^m_pW and $S^t_{p,p}B$ are studied in some areas of mathematics:

- approximation theory in high dimension
- information-based complexity
- partial differential equations and learning theory.

Isotropic Besov spaces

All the spaces are defined on $\mathbb{R}^d$.

Let $1 \le p \le \infty$, $M \in \mathbb{N}$ and $f \in L_p$. By $\Delta_h^M f$ we denote the M-th order difference of f. Here

$$\Delta_h^1 f(x) := f(x+h) - f(x), \quad \Delta_h^M f(x) := \Delta_h^1(\Delta_h^{M-1} f)(x), \quad h \in \mathbb{R}^d.$$

Definition. Let $1 \le p \le \infty$, $t > 0$ and $M > t$. Then the isotropic Besov space $B^t_{p,p}$ is the collection of all $f \in L_p$ such that

$$\|f|B^t_{p,p}\| = \|f|L_p\| + \Big(\sum_{j=0}^{\infty} 2^{jtp} \sup_{|h|<2^{-j}} \|\Delta_h^M f|L_p\|^p\Big)^{1/p} < \infty.$$

Pointwise multipliers and multiplication algebras

Definition. Let $1 \leq p \leq \infty$ and $t > 0$.

(i) A function $g \in L_1^{\mathrm{loc}}$ is called a pointwise multiplier for $B^t_{p,p}$ if for every $f \in B^t_{p,p}$ we have $gf \in B^t_{p,p}$. By $M(B^t_{p,p})$ we denote the collection of all pointwise multipliers for $B^t_{p,p}$.

(ii) The space $B^t_{p,p}$ is called a multiplication algebra (or an algebra) if for $f, g \in B^t_{p,p}$ we have $fg \in B^t_{p,p}$ and there exists $C > 0$ s.t.

$$\|fg|B^t_{p,p}\| \leq C\|f|B^t_{p,p}\| \cdot \|g|B^t_{p,p}\|.$$

Remark. We can show that if $g \in M(B^t_{p,p})$ then the mapping $f \to gf$ yields a bounded linear operator in $B^t_{p,p}$.

$B^t_{p,p}$ is an algebra

Theorem. (Peetre '70, Triebel '77) Let $1 \le p \le \infty$ and $t > 0$. Then $B^t_{p,p}$ is a multiplication algebra if and only if either $t > d/p$ or $t = d$ and $p = 1$.

Let ψ be a non-negative C_0^∞ function. We put $\psi_\mu(x) = \psi(x - \mu)$, $\mu \in \mathbb{Z}^d,\ x \in \mathbb{R}^d$ and assume that $\sum_{\mu\in\mathbb{Z}^d} \psi_\mu(x) = 1$ for $x \in \mathbb{R}^d$. We define $B^t_{p,p,\mathrm{unif}}$ as the collection of all $f \in L_1^{loc}$ s.t.

$$\|f|B^t_{p,p,\mathrm{unif}}\|_\psi = \sup_{\mu\in\mathbb{Z}^d} \|\psi_\mu f|B^t_{p,p}\| < \infty.$$

Theorem. (Peetre '76, Maz'ya and Shaposnikova '85) Let $1 \le p \le \infty$. If either $t > d/p$ or $t = d$, $p = 1$. Then

$$M(B^t_{p,p}) = B^t_{p,p,\mathrm{unif}}.$$

Mixed differences

Let $f \in L_p$, $e \subset \{1, ..., d\}$, $h \in \mathbb{R}^d$ and $m \in \mathbb{N}$. The mixed (m, e)th difference operator $\Delta_h^{m,e}$ is defined as

$$\Delta_h^{m,e} := \prod_{i \in e} \Delta_{h_i,i}^m \qquad \text{and} \qquad \Delta_h^{m,\emptyset} := \mathrm{Id}\,,$$

where $\Delta_{h_i,i}^m$ is the univariate operator applied to the i-th coordinate of f with the other variables kept fixed. We define

$$\omega_m^e(f,s)_p := \sup_{|h_i|<s_i, i\in e} \|\Delta_h^{m,e}(f,\cdot)|L_p\| \quad , \quad s \in [0,1]^d$$

in particular, $\omega_m^\emptyset(f,s)_p = \|f|L_p\|$.

Besov spaces of dominating mixed smoothness

Definition. Let $1 \le p \le \infty$, $m \in \mathbb{N}$ and $m > t > 0$. Then the Besov space of dominating mixed smoothness $S^t_{p,p}B$ is the collection of all $f \in L_p$ such that

$$\|f|S^t_{p,p}B\|^{(m)} := \sum_{e\subset\{1,\ldots,d\}} \Big(\sum_{k\in\mathbb{N}_0^d(e)} 2^{t|k|_1 p}\omega_m^e(f,2^{-k})_p^p\Big)^{1/p} < \infty\,.$$

Here $\mathbb{N}_0^d(e) = \big\{k \in \mathbb{N}_0^d : \; k_i = 0 \text{ if } i \notin e\big\}$.

Remark. (i) If $d = 1$, then we have $S^t_{p,p}B(\mathbb{R}) = B^t_{p,p}(\mathbb{R})$.

(ii) The space $S^t_{p,p}B$ has a cross-norm, i.e., if $f_i \in B^t_{p,p}(\mathbb{R})$, $i = 1, \ldots, d$, then

$$f(x) = \prod_{i=1}^d f_i(x_i) \in S^t_{p,p}B(\mathbb{R}^d) \quad \textit{and} \quad \| f\,|S^t_{p,p}B(\mathbb{R}^d)\| = \prod_{i=1}^d \| f_i\,|B^t_{p,p}(\mathbb{R})\|\,.$$

Equivalent norms

Let $\{\varphi_j\}_{j=0}^{\infty}$ be a smooth dyadic decomposition of $\mathbb{R}$, e.g., $\varphi_0 \in C_0^\infty(\mathbb{R})$ is a non-negative function with $\varphi_0 = 1$ on $[-1, 1]$, $\operatorname{supp}\varphi_0 \subset [-2, 2]$ and $\varphi_j(\xi) = \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi)$ if $j \geq 1$. For $k \in \mathbb{N}_0^d$ we define

$$\varphi_k(x) := \varphi_{k_1}(x_1) \cdot \ldots \cdot \varphi_{k_d}(x_d)$$

then we obtain the smooth dyadic decomposition of unity on $\mathbb{R}^d$.

Theorem. Let $1 \leq p \leq \infty$, $t > 0$. Then $S^t_{p,p}B$ is the collection of all temper distributions f s.t.

$$\| f \,|S^t_{p,p}B\|^\varphi := \Big(\sum_{k \in \mathbb{N}_0^d} 2^{t|k|_1 p} \, \| \mathcal{F}^{-1}[\varphi_k \, \mathcal{F} f] | L_p\|^p \Big)^{1/p} < \infty.$$

The result for $S^t_{p,p}B$

With the similar notions of pointwise multiplier and algebra as above we have:

Theorem. Let $1 \leq p \leq \infty$ and $t > 0$.

(i) The space $S^t_{p,p}B$ is a multiplication algebra if and only if either $t > 1/p$ or $p = t = 1$.

(ii) If either $t > 1/p$ or $p = t = 1$. Then we have

$$M(S^t_{p,p}B) = S^t_{p,p,\mathrm{unif}}B.$$

Remark. The conditions $t > 1/p$ or $t = p = 1$ imply that

$$S^t_{p,p}B \hookrightarrow C.$$

Proof

Proof of (i). Let $t < m \le t+1$. The main idea is to use

$$\Delta_h^{2m,e}(fg)(x) = \sum_{0\le u_i \le 2m} C_u \Delta_h^{2m-u,e} f(x + u \diamond h) \Delta_h^{u,e} g(x).$$

Proof of (ii). We employ the result in (i) and localization property of $S^t_{p,p}B$, i.e., for $1 \le p \le \infty$ and $t > 0$ we have

$$\|f|S^t_{p,p}B\| \asymp \Big(\sum_{\mu\in\mathbb{Z}^d} \|\psi_\mu f|S^t_{p,p}B\|^p \Big)^{1/p}.$$

Recall $\psi \in C_0^\infty$, $\psi_\mu(x) = \psi(x-\mu)$ and $\sum_{\mu\in\mathbb{Z}^d} \psi_\mu(x) = 1$.

Sobolev spaces of dominating mixed smoothness

Definition. Let $m \in \mathbb{N}_0$, $1 < p < \infty$. The Sobolev space of dominating mixed smoothness S^m_pW is the collection of all $f \in L_p$ s.t.

$$\|f|S^m_pW\| := \sum_{\alpha\in\mathbb{N}^d_0,\alpha_i\le m} \|D^\alpha f|L_p\| < \infty.$$

Remark. (i) If $d = 1$, then we have $S^m_pW(\mathbb{R}) = W^m_p(\mathbb{R})$.

(ii) If $m = 0$ then $S^0_pW = L_p$ is not a multiplication algebra.

(iii) If $m \in \mathbb{N}$ we have $S^m_pW \hookrightarrow C$.

Theorem. Let $m \in \mathbb{N}$ and $1 < p < \infty$. Then the space S^m_pW is an algebra and

$$M(S^m_pW) = S^m_{p,\mathrm{unif}}W.$$

We refer to Moser ('66), Strichartz ('67) for Sobolev spaces W^m_p.

Moser-type inequality

Moser ('66) showed that there exists a constant $C > 0$ s.t.

$$\|fg|W_2^m\| \leq C\big(\|f|W_2^m\| \cdot \|g|L_\infty\| + \|f|L_\infty\| \cdot \|g|W_2^m\|\big)$$

holds for all $f, g \in W_2^m \cap L_\infty$.

Theorem. (Peetre '76, Runst '86) Let A be either isotropic Sobolev spaces W_p^m (with $m \in \mathbb{N}_0, 1 < p < \infty$) or Besov spaces $B^t_{p,p}$ (with $t > 0, 1 \leq p \leq \infty$). Then there exists a constant $C > 0$ s.t.

$$\|fg|A\| \leq C\big(\|f|A\| \cdot \|g|L_\infty\| + \|f|L_\infty\| \cdot \|g|A\|\big)$$

holds for all $f, g \in A \cap L_\infty$.

Dominating mixed smoothness is different

Normally, spaces of dominating mixed smoothness ($d > 1$) have similar properties as isotropic spaces with $d = 1$. However in the case of Moser-type inequality, the picture is different.

Theorem. Let $d > 1$. By SA we denote the Sobolev spaces $S^m_p W$ (with $m \in \mathbb{N}, 1 < p < \infty$) or Besov spaces $S^t_{p,p}B$ (with $t > 1/p, 1 \le p \le \infty$). Then there is no constant $C > 0$ s.t.

$$\|fg|SA\| \le C\big(\|f|SA\| \cdot \|g|L_\infty\| + \|f|L_\infty\| \cdot \|g|SA\|\big)$$

holds for all $f, g \in SA$.

Remark. If $m = 0$ (in $S^m_p W$) or $d = 1$ we go back to the isotropic case.

Thank you very much!