On interpolation of spaces of integrable functions with respect to a vector measure

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• Let \((\Omega, \Sigma)\) be a measurable space and \(\mu\) a \(\sigma\)-finite measure on \((\Omega, \Sigma)\). If \(1 \leq p_0 \neq p_1 \leq \infty\), \(0 < \theta < 1\), \(0 < q \leq \infty\) and \(\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\),

\[
(L^{p_0}(\mu), L^{p_1}(\mu))_{\theta,q} = L^{p,q}(\mu),
\]

with equivalence of quasi-norms. In particular,

\[
(L^1(\mu), L^\infty(\mu))_{1-\frac{1}{p},p} = L^p(\mu), \quad 1 < p < \infty.
\]

• If \(m\) is a vector measure, then a similar result does not hold. Thus,

\[
(L^1(m), L^\infty(m))_{1-\frac{1}{p},p} \subsetneq L^p(m), \quad 1 < p < \infty.
\]

The inclusion \(L^\infty(m) \subseteq L^1(m)\) is weakly compact and thus, by Beauzamy’s result, \((L^1(m), L^\infty(m))_{1-\frac{1}{p},p}\) is reflexive for \(1 < p < \infty\).

**Theorem** (Beauzamy, Lecture Notes in Math. (1978))

Let \(0 < \theta < 1\) and \(1 < q < \infty\).

\((A_0, A_1)_{\theta,q}\) is reflexive \(\iff I : A_0 \cap A_1 \to A_0 + A_1\) is weakly compact.

However, \(L^p(m), p > 1\), is not reflexive whenever \(L^1(m) \neq L^1_w(m)\).


If $0 < \theta < 1$, $0 < q \leq \infty$ and $\frac{1}{p} = 1 - \theta$, it holds

$$(L^1(m), L^\infty(m))_{\theta,q} = (L^1_w(m), L^\infty(m))_{\theta,q} = L^{p,q}(\|m\|).$$

- The **Lorentz space** $\Lambda^q_{\varphi}(\|m\|)$

For $0 < q \leq \infty$ and a non-negative function $\varphi$ on $(0, \infty)$, $\Lambda^q_{\varphi}(\|m\|)$ is the space of ($m$-a.e. equivalence classes of) scalar measurable functions on $\Omega$ s.t.

$$\|f\|_{\Lambda^q_{\varphi}(\|m\|)} := \left(\int_0^\infty (\varphi(t)f^*_t(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty$$

(with the usual modification for $q = \infty$). Here $f^*_t$ is the decreasing rearrangement (w.r.t. $m$) of $f$ given by

$$f^*_t := \inf\{s > 0 : \|m\|(\{w \in \Omega : |f(w)| > s\}) \leq t\},$$

and $\|m\|(A) := \sup\{|\langle m, x^* \rangle| (A) : x^* \in B(X^*)\}$ the semivariation of $m$. If $\varphi(t) = t^{1/p}$, $\Lambda^q_{\varphi}(\|m\|) = L^{p,q}(\|m\|)$. 

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On interpolation of spaces of integrable functions with respect to a vector measure
Let \( \Omega \) be non-empty set, \( \Sigma \) a \( \sigma \)-algebra of \( \Omega \) and \( X \) a Banach space. Let \( m : \Sigma \to X \) be a countably additive vector measure.

\( L^0(m) \) denotes the space of all scalar measurable functions on \( \Omega \). \( f, g \in L^0(m) \) will be identified if are equal \( m \)-a.e., that is, whenever

\[
\|m\|(\{w \in \Omega : f(w) \neq g(w)\}) = 0.
\]

\( f \in L^0(m) \) is called **integrable** (w.r.t. \( m \)) if

i) \( f \in L^1(|\langle m, x^* \rangle|) \), for all \( x^* \in X^* \) (i.e. \( f \) is weakly integrable w.r.t. \( m \))

ii) given any \( A \in \Sigma \), there exists an element \( \int_A f dm \in X \) such that

\[
\langle \int_A f dm, x^* \rangle = \int_A f \, d \langle m, x^* \rangle, \text{ for all } x^* \in X^*.
\]

Let

\[
L^1_w(m) := \{ f : f \text{ is weakly integrable} \},
\]

\[
L^1(m) := \{ f : f \text{ is integrable} \},
\]

endowed with the norm

\[
\|f\|_1 := \sup \left\{ \int_\Omega |f| \, d \langle m, x^* \rangle : x^* \in B(X^*) \right\}.
\]
• Given $1 < p < \infty$, $f \in L^0(m)$ is said to be
  
  i) **weakly $p$-integrable (w.r.t. $m$)** if $|f|^p \in L^1_w(m)$,
  
  ii) **$p$-integrable (w.r.t. $m$)** if $|f|^p \in L^1(m)$,

Let

\[
L^p_w(m) := \{ f : f \text{ is weakly } p\text{-integrable} \},
\]

\[
L^p(m) := \{ f : f \text{ is } p\text{-integrable} \},
\]

with the norm

\[
\| f \|_p := \sup \left\{ \left( \int_{\Omega} |f|^p \, d|\langle m, x^* \rangle| \right)^{1/p} : x^* \in B(X^*) \right\}.
\]

• Some properties:

- $L^p(m)$ is a Banach lattice with order continuous norm.
- $L^p_w(m)$ is a Banach lattice with the Fatou property.
- $L^p(m)$ and $L^p_w(m)$ may not be reflexive for $p > 1$.
- If $1 < p_1 < p_2 < \infty$, then
  
  \[
  L^\infty(m) \subseteq L^{p_2}(m) \subseteq L^{p_2}_w(m) \subseteq L^{p_1}(m) \subseteq L^{p_1}_w(m) \subseteq L^1(m) \subseteq L^1_w(m).
  \]

When $m$ is a finite positive scalar measure, $\|m\|$ and $m$ coincide. But in general, for an arbitrary vector measure $m$, it holds that

$$L^p(m) \neq L^p(\|m\|) := L^{p,\|m\|}, \quad 1 \leq p < \infty.$$ 

We have the following continuous inclusions:

$$L^\infty(m) \subseteq L^{p,1}(\|m\|) \subseteq L^p(\|m\|) \subseteq L^p(m) \subseteq L^p(m) \subseteq L^p(m) \subseteq L^p,\infty(\|m\|) \subseteq L^{1,\infty}(\|m\|).$$
• A non-negative function $\rho$ defined on $\mathbb{R}^+ := (0, \infty)$ belongs to the class $Q(0, 1)$ if there exists $0 < \varepsilon < 1/2$ such that

$$\rho(t)t^{-\varepsilon} \text{ is non-decreasing (↑)} \text{ and } \rho(t)t^{-(1-\varepsilon)} \text{ is non-increasing (↓)}.$$  


• For a quasi-Banach couple $(X_0, X_1)$, the real interpolation space $(X_0, X_1)_{\rho,q}$, $\rho \in Q(0, 1)$, $0 < q \leq \infty$, consists of all $x \in X_0 + X_1$ for which

$$\|x\|_{\rho,q} := \left( \int_0^\infty \left[ \frac{K(t, x; X_0, X_1)}{\rho(t)} \right]^q \frac{dt}{t} \right)^{1/q} < \infty,$$

(with the usual modification for $q = \infty$), where the $K$-functional is defined for $t > 0$ as

$$K(t, x; X_0, X_1) = \inf \{ \|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_i \in X_i \}, x \in X_0 + X_1.$$

• When $\rho(t) = t^\theta$, $0 < \theta < 1$, we get the classical space $(X_0, X_1)_{\theta,q}$.

• It holds that

$$(L^1(\mu), L^\infty(\mu))_{\rho(t) = t^{1-\frac{1}{p}(1+|\log t|)^{-\alpha}}, q} = L^{p,q}(L)_{\alpha}(\mu).$$
Other similar classes of functions

- $B_K : \rho \in C(\mathbb{R}^+) \text{ non-decreasing such that}$
  \[ \bar{\rho}(t) = \sup_{s>0} \frac{\rho(ts)}{\rho(s)} < \infty \text{ for every } t > 0, \]
  \[ \int_0^\infty \min\{1, \frac{1}{t}\} \bar{\rho}(t) \frac{dt}{t} < \infty. \]

- $B_\psi : \rho \in C^1(\mathbb{R}^+) \text{ satisfying}$
  \[ 0 < \inf_{t>0} \frac{t\rho'(t)}{\rho(t)} \leq \sup_{t>0} \frac{t\rho'(t)}{\rho(t)} < 1. \]

- $\mathcal{P}^{+-} : \rho(t) \text{ non-decreasing, } \rho(t)/t \text{ non-increasing and}$
  \[ \bar{\rho}(t) = o(\max\{1, t\}) \text{ as } t \to 0 \text{ and } t \to \infty. \]

**Proposition** (Gustavsson, Math. Scand. (1978) / Persson, Math. Scand. (1986))

a) $B_\psi \subseteq Q(0,1) \subseteq \mathcal{P}^{+-}.$
b) $B_\psi \subseteq B_K \subseteq \mathcal{P}^{+-}.$
c) If $\rho \in \mathcal{P}^{+-}$, there exists $\varphi \in B_\psi$ such that $\rho \approx \varphi.$

- The $K$-functional for $(L^1(\|m\|), L^\infty(m))$ and $(L^{1,\infty}(\|m\|), L^\infty(m))$:

**Proposition**

For each $f \in L^1(\|m\|)$,

$$K(t, f; L^1(\|m\|), L^\infty(m)) = \int_0^\infty \min\{t, \|m\|_f(s)\} \, ds = \int_0^t f_*(s) \, ds,$$

where $\|m\|_f(t) := \|m\|(\{w \in \Omega : |f(w)| > t\})$.

**Proposition**

It holds that

$$\sup_{s>0} s \min\{t, \|m\|_f(s)\} \leq K(t, f; L^{1,\infty}(\|m\|), L^\infty(m)),$$

for all $f \in L^{1,\infty}(\|m\|)$. In particular (taking $s := f_*(t)/2$),

$$tf_*(t) \leq K(t, f; L^{1,\infty}(\|m\|), L^\infty(m)).$$

The next result follows from these estimates for the $K$-functional and weighted Hardy’s inequality for non-increasing functions:

**Theorem**

Let $\rho \in Q(0, 1)$, $0 < q \leq \infty$ and $\varphi(t) = \frac{t}{\rho(t)}$. Then,

$$(L^1(\|m\|), L^\infty(m))_{\rho,q} = (L^{1,\infty}(\|m\|), L^\infty(m))_{\rho,q} = \Lambda^q_{\varphi}(\|m\|).$$

In particular, if $0 < \theta < 1$, it holds that

$$(L^1(\|m\|), L^\infty(m))_{\theta,q} = (L^{1,\infty}(\|m\|), L^\infty(m))_{\theta,q} = L^{\frac{1}{1-\theta},q}(\|m\|).$$
Using the last theorem, reiteration and the continuous inclusions
\[ L^\infty (m) \subseteq L^r (\| m \|) \subseteq L^r (m) \subseteq L^r_w (m) \subseteq L^r,\infty (\| m \|), \quad r \geq 1, \]

**Theorem**

If \( 1 \leq p_0 \neq p_1 \leq \infty, \rho \in Q(0, 1), \varphi(t) = \frac{\frac{1}{t^{p_0}}}{\rho \left( t^{\frac{1}{p_0} - \frac{1}{p_1}} \right)} \) and \( 0 < q \leq \infty, \)
\[ (L^{p_0}(m), L^{p_1}(m))_{\rho, q} = (L^{p_0}_w(m), L^{p_1}(m))_{\rho, q} = (L^{p_0}(m), L^{p_1}_w(m))_{\rho, q} = \Lambda^q (\| m \|). \]

In particular, if \( 0 < \theta < 1 \) and \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \), it holds that
\[ (L^{p_0}(m), L^{p_1}(m))_{\theta, q} = (L^{p_0}_w(m), L^{p_1}(m))_{\theta, q} = (L^{p_0}_w(m), L^{p_1}_w(m))_{\theta, q} = L^{p, q}(\| m \|). \]

**Corollary**

For \( \rho(t) = t^{1-\frac{1}{p}}(1 + | \log t |)^{-\alpha}, 1 < p < \infty, 0 < q \leq \infty \) and \( \alpha \in \mathbb{R}, \)
\[ (L^1(m), L^\infty (m))_{\rho, q} = (L^1_w(m), L^\infty (m))_{\rho, q} = L^{p, q}(\log L)^\alpha (\| m \|). \]

When \( \varphi(t) = t^{\frac{1}{p}}(1 + | \log t |)^{\alpha}, \Lambda^q (\| m \|) = L^{p, q}(\log L)^\alpha (\| m \|), \) that can be considered the version of Lorentz-Zygmund space in the vector case.
Theorem

Let $\rho \in Q(0, 1)$, $1 \leq p < \infty$ and $0 < q_0, q, q_1 \leq \infty$.

a) If $\varphi_0 \in Q(0, 1)$,

$$(\Lambda_{\varphi_0}^{q_0}(\|m\|), L^\infty(m))_{\rho,q} = \Lambda_{\varphi}^{q}(\|m\|), \quad \varphi(t) = \frac{\varphi_0(t)}{\rho(\varphi_0(t))}.$$ 

b) If $\varphi_1 \in Q(0, 1/p)$ (i.e. $\varphi_1(t)t^{-\varepsilon} \uparrow$ and $\varphi_1(t)t^{-(\frac{1}{p}-\varepsilon)} \downarrow$ for some $0 < \varepsilon < \frac{1}{2p}$),

$$(L^p(\|m\|), \Lambda_{\varphi_1}^{q_1}(\|m\|))_{\rho,q} = \Lambda_{\varphi}^{q}(\|m\|), \quad \varphi(t) = \frac{t^{1/p}}{\rho(t^{1/p}/\varphi_1(t))}.$$ 

c) If $\varphi_i \in Q(0, 1)$, $i = 0, 1$, and $\phi := \frac{\varphi_0}{\varphi_1} \in Q(0, b)$ for some $b \in \mathbb{R}$ (i.e. $\phi(t)t^{-\varepsilon} \uparrow$ and $\phi(t)t^{-(b-\varepsilon)} \downarrow$ for some $0 < \varepsilon < \frac{b}{2}$),

$$(\Lambda_{\varphi_0}^{q_0}(\|m\|), \Lambda_{\varphi_1}^{q_1}(\|m\|))_{\rho,q} = \Lambda_{\varphi}^{q}(\|m\|), \quad \varphi(t) = \frac{\varphi_0(t)}{\rho(\varphi_0(t)/\varphi_1(t))}.$$ 


**Theorem** (Fernández, Mayoral, Naranjo and Sánchez-Pérez, Collect. Math. (2010))

Given $1 \leq p_0 \neq p_1 \leq \infty$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, it holds that

$$[L^{p_0}(m), L^{p_1}(m)][\theta] = [L^{p_0}_w(m), L^{p_1}_w(m)][\theta] = [L^{p_0}_w(m), L^{p_1}_w(m)][\theta] = L^p(m),$$

$$[L^{p_0}(m), L^{p_1}(m)][\theta] = [L^{p_0}_w(m), L^{p_1}_w(m)][\theta] = [L^{p_0}_w(m), L^{p_1}_w(m)][\theta] = L^p_w(m).$$

- Orlicz spaces $L_0^\phi(m)$ and $L_1^\phi(m)$ generalize the spaces $L_0^p(m)$ and $L_1^p(m)$, respectively. We are interested in studying if the following equalities hold:

\[
\begin{align*}
[L_0^\phi(m), L_1^\phi(m)]_{[\theta]} &= [L_0^\phi(m), L_1^\phi(m)]_{[\theta]} = [L_0^\phi(m), L_1^\phi(m)]_{[\theta]} = L_0^\phi(m), \\
[L_0^\phi(m), L_1^\phi(m)]_{[\theta]} &= [L_0^\phi(m), L_1^\phi(m)]_{[\theta]} = [L_0^\phi(m), L_1^\phi(m)]_{[\theta]} = L_1^\phi(m).
\end{align*}
\]

- Given $\phi_0, \phi_1 \in \Delta_2$, $0 < \theta < 1$, $\phi^{-1} = (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^\theta$, do the following equalities hold?

$$ [L^{\phi_0}(m), L^{\phi_1}(m)][\theta] = [L_{w}^{\phi_0}(m), L_{w}^{\phi_1}(m)][\theta] = [L_{w}^{\phi_0}(m), L_{w}^{\phi_1}(m)][\theta] = L^{\phi}(m), $$

$$ [L^{\phi_0}(m), L^{\phi_1}(m)][\theta] = [L_{w}^{\phi_0}(m), L_{w}^{\phi_1}(m)][\theta] = [L_{w}^{\phi_0}(m), L_{w}^{\phi_1}(m)][\theta] = L_{w}^{\phi}(m). $$

- An **N-function** is any function $\phi : [0, \infty) \rightarrow [0, \infty)$ which is
  - strictly increasing, $\phi(0) = 0$,
  - continuous, $\lim_{x \to 0} \frac{\phi(x)}{x} = 0$,
  - convex, $\lim_{x \to \infty} \frac{\phi(x)}{x} = \infty$.

An N-function has the **$\Delta_2$-property** (we write $\phi \in \Delta_2$) if

$$\exists C > 0 \text{ such that } \phi(2x) \leq C\phi(x) \text{ for all } x \geq 0.$$
• The **weak Orlicz space** $L^\phi_w(m)$ (w.r.t. $m$ and $\phi$) is defined as

\[
L^\phi_w(m) := \left\{ f \in L^0(m) : \|f\|_{L^\phi_w(m)} < \infty \right\},
\]

where

\[
\|f\|_{L^\phi_w(m)} := \sup \left\{ \|f\|_{L^\phi(||\langle m, x^* \rangle||)} : x^* \in B_{X^*} \right\}
= \sup_{x^* \in B_{X^*}} \inf \left\{ k > 0 : \int_\Omega \phi \left( \frac{|f|}{k} \right) d\|\langle m, x^* \rangle\| \leq 1 \right\}.
\]

$L^\phi_w(m)$ coincides with the intersection of all Orlicz $L^\phi(||\langle m, x^* \rangle||)$, $x^* \in X^*$.

• The **Orlicz space** $L^\phi(m)$ (w.r.t. $m$ and $\phi$) is defined by $S(\Sigma)L^\phi_w(m)$.

• If $\phi(x) = x^p$, $L^\phi_w(m)$ and $L^\phi(m)$ correspond to $L^p_w(m)$ and $L^p(m)$, respectively.

• The corresponding **Orlicz classes** (w.r.t. $m$ and $\phi$) are given by

\[
O^\phi_w(m) := \{ f \in L^0(m) : \phi(|f|) \in L^1_w(m) \},
\]

\[
O^\phi(m) := \{ f \in L^0(m) : \phi(|f|) \in L^1(m) \}.
\]

It holds that

$O^\phi_w(m) \subseteq L^\phi_w(m)$ and $O^\phi_w(m) \subseteq L^\phi(m)$. 
• The **weak Orlicz space** $L^\phi_w(m)$ (w.r.t. $m$ and $\phi$) is defined as

$$L^\phi_w(m) := \left\{ f \in L^0(m) : \|f\|_{L^\phi_w(m)} < \infty \right\},$$

where

$$\|f\|_{L^\phi_w(m)} := \sup \left\{ \|f\|_{L^\phi(|\langle m, x^* \rangle|)} : x^* \in B_{X^*} \right\}$$

$$= \sup_{x^* \in B_{X^*}} \inf \left\{ k > 0 : \int_\Omega \phi \left( \frac{|f|}{k} \right) d|\langle m, x^* \rangle| \leq 1 \right\}.$$

$L^\phi_w(m)$ coincides with the intersection of all Orlicz $L^\phi(|\langle m, x^* \rangle|)$, $x^* \in X^*$.

• The **Orlicz space** $L^\phi(m)$ (w.r.t. $m$ and $\phi$) is defined by $S(\Sigma)L^\phi_w(m)$.

• If $\phi(x) = x^p$, $L^\phi_w(m)$ and $L^\phi(m)$ correspond to $L^p_w(m)$ and $L^p(m)$, respect.

• The corresponding **Orlicz classes** (w.r.t. $m$ and $\phi$) are given by

$$O^\phi_w(m) := \{ f :\in L^0(m) : \phi(|f|) \in L^1_w(m) \},$$

$$O^\phi(m) := \{ f :\in L^0(m) : \phi(|f|) \in L^1(m) \}.$$

When $\phi \in \Delta_2$

$$O^\phi_w(m) = L^\phi_w(m) \text{ and } O^\phi(m) = L^\phi(m).$$
• Let \((X_0, X_1)\) be a couple of Banach lattices on the same measure space and \(0 < \theta < 1\), the \textbf{Calderón’s space} \(X_0^{1-\theta}X_1^\theta\) is

\[
X_0^{1-\theta}X_1^\theta := \{ f \in L^0 : \exists \lambda > 0, \exists f_i \in B_{X_i} \text{ s.t. } |f| \leq \lambda |f_0|^{1-\theta}|f_1|^\theta \},
\]

with the norm

\[
\|f\|_{X_0^{1-\theta}X_1^\theta} := \inf\{\lambda > 0 : |f| \leq \lambda |f_0|^{1-\theta}|f_1|^\theta, f_0 \in B_{X_0}, f_1 \in B_{X_1}\}.
\]

It holds that

\begin{itemize}
  \item [C1] \(X_0 \cap X_1 \subseteq [X_0, X_1][\theta] \subseteq X_0^{1-\theta}X_1^\theta \subseteq [X_0, X_1][\theta] \subseteq X_0 + X_1\).
  \item [C2] If \(X_0\) or \(X_1\) is order continuous, then \([X_0, X_1][\theta] = X_0^{1-\theta}X_1^\theta\).
  \item [C3] If \(X_0\) and \(X_1\) have the Fatou property then \([X_0, X_1][\theta] = X_0^{1-\theta}X_1^\theta\).
\end{itemize}
Given a Banach couple \((X_0, X_1)\) and \(0 < \theta < 1\), the **Gustavsson-Peetre space** \(\langle X_0, X_1, \theta \rangle\) is the Banach space formed by

\[
x \in X_0 + X_1 \text{ for which } \exists (x_k)_{k \in \mathbb{Z}} \subseteq X_0 \cap X_1 \text{ s.t.}
\]

a) \(x = \sum_{k \in \mathbb{Z}} x_k\), where the series converges in \(X_0 + X_1\).

b) \(\exists C > 0 \text{ s.t. for every finite subset } F \subseteq \mathbb{Z} \text{ and every subset of scalars } (\varepsilon_k)_{k \in F}, \text{ with } |\varepsilon_k| \leq 1,\)

\[
\left\| \sum_{k \in F} \frac{\varepsilon_k}{2^{k\theta}} x_k \right\|_{X_0} \leq C \quad \text{and} \quad \left\| \sum_{k \in F} \frac{\varepsilon_k}{2^{k(\theta-1)}} x_k \right\|_{X_1} \leq C.
\]

The norm considered in \(\langle X_0, X_1, \theta \rangle\) is

\[
\|x\|_{\langle X_0, X_1, \theta \rangle} = \inf\{ C > 0 : \text{taken over all } (x_k)_{k \in \mathbb{Z}} \text{ satisfying a) and b})\}.
\]

Moreover,

\[
\text{GP } \langle X_0, X_1, \theta \rangle \subseteq [X_0, X_1][\theta].
\]
Proposition

Let \( \phi_0, \phi_1 \in \Delta_2 \), \( 0 < \theta < 1 \) and let \( \phi \) be given by \( \phi^{-1} := (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta} \). Then

(1) \( L^{\phi_0}(m)^{1-\theta}L^{\phi_1}(m)^{\theta} = L^{\phi}(m) \).

(2) \( L^{\phi_0}_w(m)^{1-\theta}L^{\phi_1}_w(m)^{\theta} = L^{\phi}_w(m) \).

\( L^{\phi}(m) \) is order continuous and \( L^{\phi}_w(m) \) has the Fatou property.

Corollary

Let \( \phi_0, \phi_1 \in \Delta_2 \), \( 0 < \theta < 1 \) and \( \phi \) s.t. \( \phi^{-1} := (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta} \). It holds that

\[ [L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} = L^{\phi}(m). \]

\[ [L^{\phi_0}_w(m), L^{\phi_1}_w(m)]^{[\theta]} = L^{\phi}_w(m). \]
Some partial ordering relations between $N$-functions:

$\phi_1 \prec \phi_0$ if $\exists \varepsilon > 0$, $\exists x_0 \geq 0$ s.t. $\phi_1(x) \leq \phi_0(\varepsilon x)$, for all $x \geq x_0$.

$\phi_1 \ll \phi_0$ if $\forall \varepsilon > 0$, $\exists x_\varepsilon \geq 0$ s.t. $\phi_1(x) \leq \phi_0(\varepsilon x)$, for all $x \geq x_\varepsilon$.

**Lemma**

Let $\phi_0, \phi_1 \in \Delta_2$.

1. If $\phi_1 \prec \phi_0$, then $L_{w}^{\phi_0}(m) \subseteq L_{w}^{\phi_1}(m)$, and $L^{\phi_0}(m) \subseteq L^{\phi_1}(m)$.
2. If $\phi_1 \ll \phi_0$, then $L_{w}^{\phi_0}(m) \subseteq L^{\phi_1}(m)$.

For $\phi_1(x) := x^p$, $\phi_0(x) := x^q$, $1 < p < q$, it follows that $\phi_1 \ll \phi_0$, and therefore the well-known inclusion $L_{w}^{\phi_0}(m) \subseteq L^{\phi_1}(m)$. 

- Some partial ordering relations between $N$-functions:
  \[ \phi_1 \prec \phi_0 \text{ if } \exists \varepsilon > 0, \exists x_0 \geq 0 \text{ s.t. } \phi_1(x) \leq \phi_0(\varepsilon x), \text{ for all } x \geq x_0. \]
  \[ \phi_1 \ll \phi_0 \text{ if } \forall \varepsilon > 0, \exists x_\varepsilon \geq 0 \text{ s.t. } \phi_1(x) \leq \phi_0(\varepsilon x), \text{ for all } x \geq x_\varepsilon. \]

**Lemma**

Let $\phi_0, \phi_1 \in \Delta_2, 0 < \theta < 1$ and let $\phi$ be given by $\phi^{-1} \coloneqq (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^\theta$.

1. If $\phi_1 \prec \phi_0$, then $L_{\phi_0}(m) \subseteq L_{\phi_1}(m)$, and $L_{\phi_0}(m) \subseteq L_{\phi_1}(m)$.
2. If $\phi_1 \ll \phi_0$, then $L_{\phi_0}(m) \subseteq L_{\phi_1}(m)$.
3. If $\phi_1 \prec \phi_0$, then $\phi_1 \prec \phi \ll \phi_0$. If $\phi_1 \ll \phi_0$, then $\phi_1 \ll \phi \ll \phi_0$.

**Theorem**

Let $\phi_0, \phi_1 \in \Delta_2, 0 < \theta < 1$ and let $\phi$ be given by $\phi^{-1} \coloneqq (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^\theta$.

If $\phi_1 \ll \phi_0$, it follows that
\[
\langle L_{\phi_0}(m), L_{\phi_1}(m), \theta \rangle = L_{\phi_0}(m).
\]
\[ L^\phi_w(m) = \langle L^\phi_0(m), L^\phi_1(m), \theta \rangle \subseteq [L^\phi_0(m), L^\phi_1(m)]^{[\theta]} \]
\[ \subseteq [L^\phi_0(w)(m), L^\phi_1(w)(m)]^{[\theta]} = (L^\phi_0(w)(m))^{1-\theta}(L^\phi_1(w)(m))^\theta = L^\phi_w(m). \]

Therefore,
\[ [L^\phi_0(m), L^\phi_1(m)]^{[\theta]} = [L^\phi_0(w)(m), L^\phi_1(w)(m)]^{[\theta]} = L^\phi_w(m), \]
and, by \( L^\phi_i(m) \subseteq L^\phi_i(w)(m) \) \((i = 0, 1)\), it also holds that
\[ [L^\phi_0(m), L^\phi_1(m)]^{[\theta]} = [L^\phi_0(w)(m), L^\phi_1(w)(m)]^{[\theta]} = L^\phi_w(m). \]

This gives (i) in the following theorem.

**Theorem**

Let \( \phi_0, \phi_1 \in \Delta_2, 0 < \theta < 1 \) and let \( \phi \) be given by \( \phi^{-1} := (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^\theta. \)
If \( \phi_1 \precsim \phi_0 \), then

(i) \([L^\phi_0(m), L^\phi_1(m)]^{[\theta]} = [L^\phi_0(w)(m), L^\phi_1(w)(m)]^{[\theta]} = [L^\phi_0(w)(m), L^\phi_1(w)(m)]^{[\theta]} = L^\phi_w(m).\]

(ii) \([L^\phi_0(w)(m), L^\phi_1(w)(m)]^{[\theta]} = [L^\phi_0(w)(m), L^\phi_1(w)(m)]^{[\theta]} = [L^\phi_0(w)(m), L^\phi_1(w)(m)]^{[\theta]} = L^\phi_w(m).\)
Some references