Trace and extension theorems

for BV and Sobolev functions in metric spaces

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Trace and extension theorems

Boundary value problems seek to find a solution of a PDE subject a prescribed condition on the behavior at the boundary of the domain.

Trace theorems

- Given a domain Ω and a function u of a "Sobolev" class on Ω, describe the meaning $u|_{\partial\Omega}$;
- What properties does $u|_{\partial\Omega}$ possess in terms of integrability or smoothness?
- Is the mapping $T : u \mapsto u|_{\partial\Omega}$ a bounded operator between some function spaces?

Extension theorems

- Given a domain Ω and a function f on $\partial\Omega$, is it possible to find a function u a "Sobolev" class on Ω, such that $u|_{\partial\Omega} = f$;
- What qualities does f need to have to be able to find u?
- Is the mapping $\mathrm{Ext} : f \mapsto u$ a bounded operator between some function spaces?

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BV and Sobolev spaces in $\mathbf{R}^n$

Functions of a Sobolev space are "well-behaved" as they are weakly differentiable and both the function and its distributional gradient ∇u, which is defined via

$$\int_\Omega u \nabla \varphi \, dx = -\int_\Omega \varphi \nabla u \, dx \quad \text{for every } \varphi \in \mathcal{C}_c^\infty(\Omega),$$

are controlled by the L^p norm.

Definition

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega, \mathbf{R}^n)\},$$
$$BV(\Omega) = \{u \in L^1(\Omega) : Du \in \mathcal{M}(\Omega, \mathbf{R}^n)\} \supset W^{1,1}(\Omega).$$

Remark: $\|u\|_{W^{1,p}} = 0$ iff $u = 0$ a.e.

Problem: $|\partial\Omega| = 0$ for every "decent" domain $\Omega \subset \mathbf{R}^n$

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Trace and extension theorems in $\mathbf{R}^n$

Theorem (Gagliardo, 1957)

- *The trace space of* $W^{1,1}(\mathbf{R}^{n+1}_+)$ *is* $L^1(\mathbf{R}^n)$.
- *The trace space of* $W^{1,p}(\mathbf{R}^{n+1}_+)$ *is* $W^{1/p',p}(\mathbf{R}^n)$ *whenever* $p > 1$.

More precisely:

- There is a (surjective) bounded linear trace operator

$$T : W^{1,1}(\mathbf{R}^{n+1}_+) \to L^1(\mathbf{R}^n).$$

- There is a (surjective) bounded linear trace operator

$$T : W^{1,p}(\mathbf{R}^{n+1}_+) \to W^{1/p',p}(\mathbf{R}^n).$$

- There is a bounded linear extension operator

$$\mathrm{Ext} : W^{1/p',p}(\mathbf{R}^n) \to W^{1,p}(\mathbf{R}^{n+1}_+)$$

such that $T(\mathrm{Ext}\, f) = f$ a.e. in $\mathbf{R}^n$.

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Trace and extension theorems in $\mathbf{R}^n$

Remarks

- Gagliardo's result extends to BV functions:
  The trace space of $BV(\mathbf{R}^{n+1}_+)$ is $L^1(\mathbf{R}^n)$.
- Analogous statements hold also for $\Omega \subset \mathbf{R}^n$ provided that its boundary $\partial\Omega$ is Lipschitz.

Question

Is there a bounded linear extension operator $L^1(\mathbf{R}^n) \to BV(\mathbf{R}^{n+1}_+)$?

No!

Peetre (1979) proved that the extension operator $L^1(\mathbf{R}^n) \to BV(\mathbf{R}^{n+1}_+)$ cannot be linear.

If it were linear and bounded, then the Dirac measure would be absolutely continuous with respect to the Lebesgue measure.

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What about more complex domains?

The trace & extension problems can be studied in the setting of metric spaces. These provide us with a framework for:

- subsets of (weighted) $\mathbf{R}^n$;
- fractal sets;
- Carnot–Carathéodory spaces, Heisenberg groups;
- Riemannian manifolds.

Caveat 1: The definition of the distributional gradient uses the linear structure of $\mathbf{R}^n$, which is missing in metric spaces.

Let's use upper gradients (and/or Poincaré inequality)!

Caveat 2: What measure should be used on $\partial\Omega$?

Let's use Hausdorff codimension θ measure!

Upper gradients in $\mathbf{R}^n$

Consider a smooth function $u : \Omega \to \mathbf{R}$ and a smooth curve $\gamma : [0, l_\gamma] \to \Omega$. Then the function $u \circ \gamma : [0, l_\gamma] \to \mathbf{R}$ is smooth and hence the Newton–Leibniz formula applies, i.e.,

$$u(\gamma(l_\gamma)) - u(\gamma(0)) = \int_0^{l_\gamma} (u \circ \gamma)'(t)\, dt = \int_0^{l_\gamma} \nabla u(\gamma(t)) \cdot \gamma'(t)\, dt\,.$$

Consequently,

$$|u(\gamma(l_\gamma)) - u(\gamma(0))| \le \int_0^{l_\gamma} |\nabla u(\gamma(t))||\gamma'(t)|\, dt = \int_\gamma |\nabla u|\, ds.$$

Upper gradients in a metric space X

Definition (J. Heinonen, P. Koskela, 1996)

A Borel function $g : X \to [0, \infty]$ is an *upper gradient* of $u : X \to \overline{\mathbf{R}}$ if

$$|u(\gamma(l_\gamma)) - u(\gamma(0))| \le \int_\gamma g\, ds$$

for every rectifiable curve $\gamma : [0, l_\gamma] \to X$.

Remark: An upper gradient is by no means given uniquely.

Definition (N. Shanmugalingam, 1999)

$$N^{1,p}(X) = \{u \in L^p(X) : \text{there is an upper gradient } g \in L^p(X) \text{ of } u\}$$

$$\|u\|_{N^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)} .$$

Remark: $N^{1,p}(\mathbf{R}^n)/_{= \text{a.e.}} = W^{1,p}(\mathbf{R}^n)$ for all $p \in [1, \infty]$.

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BV in a metric space X

Definition

The *total variation* of $u \in L^1_{\mathrm{loc}}(X)$ is

$$\|Du\|(X) = \inf\{\liminf_{i\to\infty} \|g_{u_i}\|_{L^1(X)} : u_i \in \mathrm{Lip}_{\mathrm{loc}}(X), u_i \to u \in L^1_{\mathrm{loc}}(X)\}.$$

Analogously, we can define $\|Du\|(U)$ for any open set $U \subset X$.

M. Miranda Jr. (2003) showed that $U \mapsto \|Du\|(U)$ gives a Radon measure provided that $\|Du\|(X) < \infty$.

Definition

$$\|u\|_{BV(X)} = \|u\|_{L^1(X)} + \|Du\|(X).$$

Remark: In $\mathbf{R}^n$, the metric notion of BV functions coincides with the usual BV functions, where the vector-valued Radon measure Du is obtained via integration by parts.

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Hausdorff measure

Euclidean setting

Let the ambient space be $\mathbf{R}^n$, and we're defining $\mathcal{H}^{n-1}(E)$:

$$\mathcal{H}^{n-1}_R(E) = \inf\Big\{\sum_i c_{n-1}\,\mathrm{rad}(B_i)^{n-1} : E \subset \bigcup_i B_i \quad \& \quad \mathrm{rad}(B_i) < R\Big\}$$

$$\mathcal{H}^{n-1}(E) = \lim_{R\to 0^+} H^{n-1}_R(E) = \sup_{R>0} H^{n-1}_R(E)$$

Observe that

$$\mathrm{rad}(B_i)^{n-1} = \frac{\mathrm{rad}(B_i)^n}{\mathrm{rad}(B_i)^1} = c_n \frac{|B_i|}{\mathrm{rad}(B_i)^1}$$

Definition (Hausdorff measure of codimension $\theta \geq 0$)

$$\mathcal{H}_\theta(E) = \sup_{R>0} \inf\Big\{\sum_i \frac{\mu(B_i)}{\mathrm{rad}(B_i)^\theta} : E \subset \bigcup_i B_i \quad \& \quad \mathrm{rad}(B_i) < R\Big\}$$

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Trace theorems in metric spaces

Setting

- (X, d, μ) is a metric space endowed with a doubling measure μ.
- Ω admits a 1-Poincaré inequality

$$\fint_B |u - u_B| \, d\mu \lesssim \operatorname{rad}(B) \frac{\|Du\|(\lambda B)}{\mu(\lambda B)}.$$

- $\Omega \subset X$ is a domain that satisfies the measure density condition

$$\mu(B(z,r) \cap \Omega) \approx \mu(B(z,r)), \quad z \in \Omega, r \le \operatorname{diam}(\Omega).$$

- $\partial\Omega$ is Ahlfors codimension-1 regular

$$\mathcal{H}_1(B(x,r) \cap \partial\Omega) \approx \frac{\mu(B(x,r))}{r}, \quad x \in \partial\Omega, r \le \operatorname{diam}(\Omega).$$

Theorem (P. Lahti, N. Shanmugalingam, 2015)

There is a bounded linear trace $T : BV(\Omega) \to L^1(\partial\Omega)$.

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Extension theorems in metric spaces

Besov spaces, Fractional Sobolev spaces (for a bounded $E \subset X$)

$$\|u\|^p_{B^\alpha_{p,p}(E)} = \|u\|^p_{L^p(E)} + \int_0^1 \int_E \fint_{B(x,t)\cap E} \frac{|u(y)-u(x)|^p}{t^{\alpha p+1}} d\nu(y)\, d\nu(x)\, dt$$

$$\approx \|u\|^p_{L^p(E)} + \int_E \int_E \frac{|u(y)-u(x)|^p}{\mu(B(x,\mathrm{d}(x,y))\cap E)\,\mathrm{d}(x,y)^{\alpha p}} d\nu(y)\, d\nu(x)$$

Theorem (L. M., N. Shanmugalingam, M. Snipes, 2015)

Assume that μ is doubling and

$$\frac{\mu(B(z,r)\cap\Omega)}{r} \le C\mathcal{H}_1(B(z,r)\cap\partial\Omega) \quad \text{for every } z\in\partial\Omega, r\in(0,R),$$

$$\frac{\mu(B(z,r)\cap\Omega)}{r} \ge c_z\mathcal{H}_1(B(z,r)\cap\partial\Omega) \quad \text{for } \mathcal{H}_1\text{-a.e. } z\in\partial\Omega, r\in(0,R_z),$$

Then, there is a ***bounded linear extension*** *operator*
$\mathrm{Ext} : B^0_{1,1}(\partial\Omega) \to BV(\Omega)$ *such that* $T(\mathrm{Ext}\, f) = f$ $\mathcal{H}_1$*-a.e. in* $\partial\Omega$.

Remark: $BV(E) \subsetneq B^0_{1,1}(E) \subsetneq L^1(E)$.

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Extension theorems in metric spaces

Theorem ($p > 1$)

Setting as in the previous slide. Then, there is a ***bounded linear extension*** *operator* $\mathrm{Ext} : B^{1/p'}_{p,p}(\partial\Omega) \to N^{1,p}(\Omega)$ *such that* $T(\mathrm{Ext}\, f) = f$ $\mathcal{H}_1$*-a.e. in* $\partial\Omega$*.*

Comparison with Euclidean case

- For $p > 1$, we have exactly the same extension result as for domains in $\mathbf{R}^n$.
- For $p = 1$, the linear extension seems to be a new result even in the Euclidean setting.
- What about L^1-boundary data for $p = 1$?

Theorem (L. M., N. Shanmugalingam, M. Snipes, 2015)

Setting as before. Then, there is a bounded ***non-linear*** *extension operator* $E : L^1(\partial\Omega) \to BV(\Omega)$ *such that* $T(Ef) = f$ $\mathcal{H}_1$*-a.e.*

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Domains with a thick or a thin boundary

Setting

- (X, d, μ) is a metric space endowed with a doubling measure μ.
- $\Omega \subset X$ is a domain that satisfies the measure density condition

$$\mu(B(z,r) \cap \Omega) \approx \mu(B(z,r)), \quad z \in \Omega, r \le \operatorname{diam}(\Omega).$$

- $\partial\Omega$ is Ahlfors codimension-θ regular

$$\mathcal{H}_\theta(B(x,r) \cap \partial\Omega) \approx \frac{\mu(B(x,r))}{r^\theta}, \quad x \in \partial\Omega, r \le \operatorname{diam}(\Omega)$$

for some $\theta > 0$.

Example

- $X = \mathbf{R}^n$ and Ω is a fractal set, e.g., the von Koch snowflake
- $X = Z \times [0, \infty)$ endowed with a product measure, where $\mathbf{R}_0^+$ is given a weighted Lebesgue measure $\Omega = Z \times (0, \infty)$, $\partial\Omega = X \times \{0\}$.

Domains with a thick or a thin boundary
Trace theorems

Proposition (A. Gogatishvili, P. Koskela, N. Shanmugalingam, 2010)

Assume that both μ and $\mathcal{H}_\theta$ are Ahlfors regular, and that X admits a p-Poincaré inequality. If Ω is an $N^{1,p}$-extension domain, then there is a linear trace operator $T : N^{1,p}(\Omega) \to B^{\alpha(q)}_{q,q}(\partial\Omega)$, which is bounded for every $q < p^*$.

Theorem (L. M., 2016)

Let the setting be as in the previous slide. In addition, assume that Ω admits a p-Poincaré inequality and $p > \theta$. Then,

$$Tu(z) := \limsup_{r\to 0^+} \fint_{B(z,r)\cap\Omega} u(x)\, d\mu(x), \qquad z \in \partial\Omega.$$

is a linear trace operator $T : N^{1,p}(\Omega) \to B^{\alpha}_{p(\alpha),p(\alpha)}(\partial\Omega)$, which is bounded for every $0 \le \alpha < 1 - \theta/p$, where $p(\alpha) \ge p$.

If Ω is a uniform domain, then $T : N^{1,p}(\Omega) \to B^{1-\theta/p}_{p,p}(\partial\Omega)$.

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Domains with a thick or a thin boundary
Extension theorems

Theorem (L. M., 2016)

Let $0 < \theta \le p$ for some $p \ge 1$. Assume that μ is doubling and

$$\frac{\mu(B(z,r)\cap\Omega)}{r^\theta} \le C\mathcal{H}_\theta(B(z,r)\cap\partial\Omega) \quad \text{for every } z \in \partial\Omega, r \in (0,R),$$

$$\frac{\mu(B(z,r)\cap\Omega)}{r^\theta} \ge C_z\mathcal{H}_\theta(B(z,r)\cap\partial\Omega) \quad \text{for a.e. } z \in \partial\Omega, r \in (0,R_z),$$

Then, there is a ***bounded linear extension*** *operator*
$\mathrm{Ext} : B^{1-\theta/p}_{p,p}(\partial\Omega) \to N^{1,p}(\Omega)$ *such that*

$$T(\mathrm{Ext}\, f) = f \quad \mathcal{H}_\theta\text{-a.e. in } \partial\Omega.$$

Corollary of the trace theorem

If Ω has a thick boundary, i.e, $\theta < 1$, then there is no extension operator mapping $L^1(\partial\Omega)$ to $BV(\Omega)$.

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Thank you for your attention!

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