On the asymptotic mean value property for planar p-harmonic functions

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The p-laplacian

p-harmonic functions

$\Omega \subset \mathbb{R}^d$, $1 < p < \infty$. $u \in W^{1,p}_{loc}(\Omega)$ is ***p*-harmonic** in Ω iff

$$div(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{(weak sense)}$$

- Euler-Lagrange equation associated to minimization of p-norm.
- p-harmonic functions solve the Dirichlet problem in nice domains.
- p-harmonic functions are $C^{1,\alpha}_{loc}$ for some $0 < \alpha < 1$ (Uraltseva 68, Lewis 77).
- Quariregular mappings, Calculus of Variations, Game theory, nonlinear elasticity...

The p-laplacian

Two relevant questions

- What is the stochastic process associated to the p-laplacian?
- What is the appropriate **mean value property** associated to the p-laplacian?

(Peres-Schramm-Sheffield-Wilson 2009, Peres-Sheffield 2008)

The mean value property and the laplacian

$\Omega \subset \mathbb{R}^d$, $u \in C(\Omega)$.

Two classical results

- (Direct) u is harmonic in Ω iff $u(x) = \fint_{B(x,r)} u$ for each $x \in \Omega$ and each $0 < r < dist(x, \partial\Omega)$. (Gauss, Koebe).
- (Asymptotic) u is harmonic in Ω iff the asymptotic MVP

$$u(x) = \fint_{B(x,r)} u + o(r^2)$$

holds at each $x \in \Omega$. (Blaschke, Privalov, Zaremba)

The p-laplacian: preliminary computations

$\triangle_p$ as a combination of $\triangle$ and $\triangle_\infty$

$\Omega \subset \mathbb{R}^d$, $u \in C^2(\Omega)$, $\nabla u \neq 0$. Then

$$\triangle_p u = div(|\nabla u|^{p-2}\nabla u) = \left[(p-2)\frac{\triangle_\infty u}{|\nabla u|^2} + \triangle u\right] |\nabla u|^{p-2}$$

where

$$\triangle_\infty u = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i,x_j} = < (Hu)\nabla u, \nabla u >$$

is the so called infinity laplacian of u.

Manipulating Taylor

$\Omega \subset \mathbb{R}^d$, $u \in C^2(\Omega)$, $x \in \Omega$, $\nabla u(x) \neq 0$, $|h| \leq r$

$$u(x+h) = u(x) + < \nabla u(x), h > + \frac{1}{2} < Hu(x)h, h > + o(r^2)$$

Choose $h = \pm r \dfrac{\nabla u(x)}{|\nabla u(x)|}$. Then:

$$u(x \pm r \frac{\nabla u(x)}{|\nabla u(x)|}) = u(x) \pm r|\nabla u(x)| + \frac{r^2 \triangle_\infty u(x)}{2|\nabla u(x)|^2} + o(r^2)$$

where

$$\triangle_\infty u =< (Hu)\nabla u, \nabla u >= \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i,x_j}$$

Averaging Taylor

Idea: Use Taylor to average u over $B(x, r)$ in two different ways:

1. $\frac{1}{2}\big(\sup_{B(x,r)} u + \inf_{B(x,r)} u \big)$ (mid-range average).
2. $⨍_{B(x,r)} u$ (usual average).

Averaging Taylor

Mid-range averages

Suppose

$$\sup_{B(x,r)} u \approx u\Big(x + r\frac{\nabla u(x)}{|\nabla u(x)|}\Big)\,, \qquad \inf_{B(x,r)} u \approx u\Big(x - r\frac{\nabla u(x)}{|\nabla u(x)|}\Big)$$

Then:

$$\frac{1}{2}\Big(\sup_{B(x,r)} u + \inf_{B(x,r)} u\Big) - u(x) = r^2\,\frac{\triangle_\infty u(x)}{2|\nabla u(x)|^2} + o(r^2)$$

Averaging Taylor

Usual averages

$$\fint_{B(x,r)} u - u(x) = r^2 \frac{\triangle u(x)}{2(d+2)} + o(r^2)$$

Averaging Taylor

Lemma

$\Omega \subset \mathbb{R}^d$, $u \in C^2(\Omega)$, $x \in \Omega$, $\nabla u(x) \neq 0$

1. $$\fint_{B(x,r)} u - u(x) = r^2 \frac{\triangle u(x)}{2(d+2)} + o(r^2)$$
2. $$\frac{1}{2}\Big(\sup_{B(x,r)} u + \inf_{B(x,r)} u \Big) - u(x) = r^2 \frac{\triangle_\infty u(x)}{2|\nabla u(x)|^2} + o(r^2)$$

Recall that (for smooth u with non-vanishing gradient):

$$\triangle_p u = div(|\nabla u|^{p-2} \nabla u) = \left[(p-2) \frac{\triangle_\infty u}{|\nabla u|^2} + \triangle u \right] |\nabla u|^{p-2}$$

The asymptotic p-mean value property

Corollary (Asymptotic p-MVP, smooth case)

$\Omega \subset \mathbb{R}^d$, $u \in C^2(\Omega)$, $x \in \Omega$, $\nabla u(x) \neq 0$. Then

$$u(x) = \frac{p-2}{p+d}\frac{1}{2}\Big(\sup_{B(x,r)} u + \inf_{B(x,r)} u\Big) + \frac{d+2}{p+d} \fint_{B(x,r)} u + $$

$$+ r^2 \frac{|\nabla u(x)|^{2-p}}{2(p+d)} \triangle_p u(x) + o(r^2)$$

as $r \to 0$. In particular,

$$\boxed{u(x) = \frac{p-2}{p+d}\frac{1}{2}\Big(\sup_{B(x,r)} u + \inf_{B(x,r)} u\Big) + \frac{d+2}{p+d} \fint_{B(x,r)} u + o(r^2)}$$

as $r \to 0$ if and only if $\triangle_p u(x) = 0$.

The asymptotic p-mean value property

The Asymptotic p- Mean Value Property (p-AMVP)

$\Omega \subset \mathbb{R}^d$, $u \in C(\Omega)$, $1 < p < \infty$. We say that u satisfies the p-AMVP at $x \in \Omega$ if

$$u(x) = \frac{p-2}{p+d}\frac{1}{2}\Big(\sup_{B(x,r)} u + \inf_{B(x,r)} u \Big) + \frac{d+2}{p+d} \fint_{B(x,r)} u + o(r^2)$$

as $r \to 0$.

The asymptotic p-mean value property

- $u \in C^2(\Omega)$, $\nabla u(x) \neq 0$. Then $\triangle_p u(x) = 0$ iff u satisfies the p-AMVP at x. In particular, p-harmonic functions satisfy p-AMVP at non-critical points (C^∞ outside critical points).
- Difficulty: p- harmonic functions are only $C^{1,\alpha}_{loc}$ for some $0 < \alpha < 1$ but not C^2 in general.
- $u \in C(\Omega)$ satisfies the p-AMVP $\Rightarrow$ u is p-harmonic. (Manfredi-Parviainen-Rossi 2009).
- The other implication is harder: problem at critical points.

The asymptotic p-mean value property

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Main question

Let u be p-harmonic in Ω and $x \in \Omega$ with $\nabla u(x) = 0$. Does u satisfy the p-AMVP at x?

The p-AMVP in the plane

Theorem

Let $\Omega \subset \mathbb{R}^2$ and let u be p-harmonic in Ω. Then u satisfies the p-AMVP in Ω.

- $1 < p < p_0 = 9,52...$ (Lindqvist-Manfredi, 2016).
- $1 < p < \infty$ (Arroyo-Llorente, 2016).

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Steps of the proof:

1. Hodograph method.
2. Power series expansion techniques (Iwaniec-Manfredi, 1989).
3. Approximation: replace the original p-harmonic function u by a simpler one $\mathfrak{U}$ (Lindqvist-Manfredi, 2016).

p-harmonic functions in the plane

The setting

- $\Omega \subset \mathbb{R}^2$, $u : \Omega \to \mathbb{R}$ p-harmonic in Ω.
- Let $f(z) = u_z = \frac{1}{2}(u_x - iu_y)$ be the **complex gradient** of u. Then f is a quasiregular mapping satisfying

$$f_{\bar{z}} = \frac{2-p}{2p}\left(\frac{\overline{f}}{f}\, f_z + \frac{f}{\overline{f}}\, \overline{f_z}\right)$$

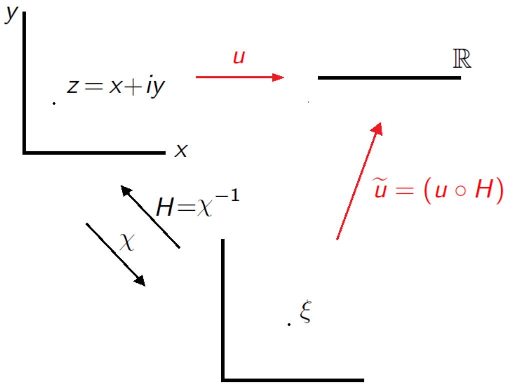
  In particular the critical points of u are isolated.
- Assume $0 \in \Omega$ is an (isolated) critical point of u of order n. Then we can represent

$$f(z) = (\chi(z))^n$$

  with χ quasiconformal locally around 0, $\chi(0) = 0$. (Stoilow).

The hodograph plane

- $f(z) = u_z = (\chi(z))^n$, complex gradient of u.
- $H = \chi^{-1}$, $\xi = \chi(z) \Leftrightarrow z = H(\xi)$. The ξ-plane is called the **hodograph plane**.
- Definim $\widetilde{u} = u \circ H$.



The hodograph plane

Advantadge of hodograph coordinates (Bers-Lavrentiev)

Whereas χ satisfies the (nonlinear) equation

$$\chi_{\bar{z}} = \frac{2-p}{2p}\left(\frac{\overline{\chi}^n}{\chi^n}\,\chi_z + \frac{\chi}{\overline{\chi}}\,\overline{\chi_z}\right)$$

it turns out that $H = \chi^{-1}$ satisfies the linear equation

$$H_{\bar{\xi}} = \frac{p-2}{2p}\left(\frac{\xi}{\bar{\xi}}\,H_\xi + \frac{\bar{\xi}^n}{\xi^n}\,\overline{H_\xi}\right)$$

Series expansion for H and $\widetilde{u}$

The following power series expansions hold for H and $\widetilde{u}$ locally around 0 (Iwaniec-Manfredi,89):

$$z = H(\xi) = \sum_{k=n+1}^{\infty} \left[A_k \left(\frac{\xi}{|\xi|} \right)^k + \varepsilon_k \overline{A_k} \left(\frac{\bar{\xi}}{|\xi|} \right)^k \right] \left(\frac{\xi}{|\xi|} \right)^{-n} |\xi|^{\lambda_k}$$

$$\widetilde{u}(\xi) = (u \circ H)(\xi) = 4 \sum_{k=n+1}^{\infty} \mu_k |\xi|^{n+\lambda_k} \mathfrak{Re} \left\{ A_k \left(\frac{\xi}{|\xi|} \right)^k \right\}$$

where $A_k \in \mathbb{C}$, $A_{n+1} \neq 0$, $\sum_{k=n+1}^{\infty} k|A_k|^2 < \infty$ and

$$\lambda_k = \frac{1}{2} \left(\sqrt{4k^2(p-1) + n^2(p-2)^2} - np \right)$$

$$\varepsilon_k = \frac{\lambda_k + n - k}{\lambda_k + n + k} \; , \quad \mu_k = \frac{\lambda_k}{\lambda_k + n + k}$$

Approximation

Let $\mathcal{A}$ and $\widetilde{\mathfrak{U}}$ be the first terms of the power series representations of H and $\widetilde{u}$ respectively.

$$H(\xi) = \sum_{k=n+1}^{\infty} \left[A_k \left(\frac{\xi}{|\xi|}\right)^k + \varepsilon_k \overline{A_k} \left(\frac{\bar{\xi}}{|\xi|}\right)^k \right] \left(\frac{\xi}{|\xi|}\right)^{-n} |\xi|^{\lambda_k}$$

$$\mathcal{A}(\xi) = \left[A_{n+1} \left(\frac{\xi}{|\xi|}\right)^{n+1} + \varepsilon_{n+1} \overline{A_{n+1}} \left(\frac{\bar{\xi}}{|\xi|}\right)^{n+1} \right] \left(\frac{\xi}{|\xi|}\right)^{-n} |\xi|^{\lambda_{n+1}}$$

$$\widetilde{u}(\xi) = 4 \sum_{k=n+1}^{\infty} \mu_k |\xi|^{n+\lambda_k} \mathfrak{Re} \left\{ A_k \left(\frac{\xi}{|\xi|}\right)^k \right\}$$

$$\widetilde{\mathfrak{U}}(\xi) = 4\, \mu_{n+1} |\xi|^{n+\lambda_{n+1}} \mathfrak{Re} \left\{ A_{n+1} \left(\frac{\xi}{|\xi|}\right)^{n+1} \right\}$$

Approximation

Put $\mathfrak{U} = \widetilde{\mathfrak{U}} \circ \mathcal{A}^{-1}$ and recall that $\widetilde{u} = u \circ H$.

Basic approximation Lemma

$$u(z) = \mathfrak{U}(z) + O\big(|z|^{\frac{n+\lambda_{n+2}}{\lambda_{n+1}}}\big) \quad (z \to 0)$$

Let ξ, ζ in the hodograph plane s.t. $z = H(\xi) = \mathcal{A}(\zeta)$. Then

1. $u(z) = \widetilde{u}(\xi),\ \mathfrak{U}(z) = \widetilde{\mathfrak{U}}(\zeta)$.
2. $|\xi| \approx |\zeta| \approx |z|^{1/\lambda_{n+1}}$.
3. $|\widetilde{u}(\xi) - \widetilde{\mathfrak{U}}(\zeta)| = O\big(|\xi|^{n+\lambda_{n+2}}\big),\ |\widetilde{\mathfrak{U}}(\xi) - \widetilde{\mathfrak{U}}(\zeta)| = O\big(|\xi|^{n+\lambda_{n+2}}\big)$.
4. $u(z) - \mathfrak{U}(z) = \widetilde{u}(\xi) - \widetilde{\mathfrak{U}}(\xi) + \widetilde{\mathfrak{U}}(\xi) - \widetilde{\mathfrak{U}}(\zeta) = O\big(|z|^{\frac{n+\lambda_{n+2}}{\lambda_{n+1}}}\big)$.

Proof of the theorem

u p-harmonic, $u(0) = 0$, u has a critical point of order n at 0.

Goal: check the p-AMVP at 0

Is it true that

$$\frac{p-2}{p+2} \cdot \frac{1}{2}\Big(\sup_{D(0,r)} u + \inf_{D(0,r)} u\Big) + \frac{4}{p+2} ⨍_{D(0,r)} u = o(r^2)$$

as $r \to 0$?

Strategy: replace u by $\mathfrak{U}$.

Proof of the theorem

- From the symmetry properties of $\widetilde{\mathfrak{U}}$ and $\mathcal{A}$, it follows

$$\sup_{D_r} \mathfrak{U} + \inf_{D_r} \mathfrak{U} = \int_{D_r} \mathfrak{U} = 0$$

for small r, where $D_r = D(0, r)$.

- Since $u(z) = \mathfrak{U}(z) + O(|z|^{\frac{n+\lambda_{n+2}}{\lambda_{n+1}}})$ as $z \to 0$, the p-AMVP at 0 would follow if

$$\frac{n + \lambda_{n+2}}{\lambda_{n+1}} > 2$$

.

Proof of the theorem

Lemma

The inequality

$$\frac{n + \lambda_{n+2}}{\lambda_{n+1}} > 2$$

hols for each $1 < p < \infty$ and each $n \geq 1$.

Remarks

- Lindqvist and Manfredi get the estimate

$$u(z) = \mathfrak{U}(z) + O\left(|z|^{1+\frac{n-1+\lambda_{n+2}}{\lambda_{n+1}^2}}\right)$$

but the inequality

$$1 + \frac{n-1+\lambda_{n+2}}{\lambda_{n+1}^2} > 2$$

only holds for $1 < p < p_0 = 9.52...$

- What if $d > 2$?