FSDONA 9

A. Yu. Golovko

Steklov Mathematical Institute of RAS

Embedding theorems of anisotropic Sobolev spaces on irregular domains

Praha, 9 Jule 2016

In 1938, for bounded domains $G \subset R^n$ satisfying the cone condition, S.L. Sobolev established an embedding theorem $W_p^s(G) \subset L_q(G)$ characterized by the inequality

$$\|f\|_{L_q(G)} \leqslant C\|f\|_{W_p^s(G)} = C\left(\sum_{|a|=s} \|D^\alpha f\|_{L_p(G)} + \|f\|_{L_p(G)}\right),$$

where $1 < p < q < \infty$ and $s \in \mathbb{N}$, provided that

$$s - \frac{n}{p} + \frac{n}{q} \geqslant 0.$$

Later, this theorem was extended to more general classes of domains.

We will use the following notations:

$$\rho(x) = \mathrm{dist}(x, \mathbb{R}^n \setminus G), \text{ where } G \subset \mathbb{R}^n \text{ is domain}$$
$$B(x, R) = \{y : |y - x| < R\}.$$

Definition. For $\sigma \geqslant 1$, a domain $G \subset \mathbb{R}^n$ is called the domain with the flexible σ-cone condition if, for some $T_0 > 0$, $\varkappa > 0$ and any $x \in G$, there exists a piecewise smooth path $\gamma : [0, T_0] \to G, \gamma(0) = x, |\gamma'| \leqslant 1$ almost everywhere, such that $\rho(\gamma(t)) \geqslant \varkappa t^\sigma$ for $0 < t = T_0$.

In 2001, O.V. Besov proved this theorem for domains with the flexible σ-cone condition provided that the following expression holds:

$$s - \frac{\sigma(n-1)+1}{p} + \frac{n}{q} \geqslant 0.$$

In 2010, O.V. Besov extended this embedding theorem to the case of norms of a more general form (which includes the sum of norms of only part of the generalized partial derivatives of order s).

Theorem 1 (G., 2015). Let G be a domain with the flexible σ-cone condition, $1 \leqslant p_j, q, r < \infty$, $p_j < q$, $r \leqslant q$, $p_j > 1$, $s_j \in \mathbb{N}$, $l \in \mathbb{Z}_+$, $l < s_j$ for $j = \overline{1, n}$. Then the estimate

$$\sum_{|\alpha|=l} \|D^\alpha f\|_{L_q(G)} \leqslant C \left(\sum_{j=1}^{n} \left\| \frac{\partial^{s_j} f}{\partial x_j^{s_j}} \right\|_{L_{p_j}(G)} + \|f\|_{L_r(G)} \right) \qquad (1)$$

is valid for functions f with finite right-hand side provided that the following expression holds for all $j = \overline{1, n}$:

$$l - \frac{n}{q} \leqslant s_j - (\sigma - 1) \sum_{i=1, i \neq j}^{n} (s_i - 1) - \frac{\sigma(n-1)+1}{p_j}. \qquad (2)$$

In isotropic case ($s_j = s$, $p_j = p$ for $j = \overline{1, n}$) theorem 1 coincides with embedding theorem which O.V. Besov proved in 2010.

This theorem is sharp in the class of domains with the flexible σ-cone condition.

In 1959, E. Gagliardo and L. Nirenberg established the following inequality for domains $G \subset \mathbb{R}^n$ with smooth boundary:

$$\sum_{|\alpha|=l} \|D^\alpha f\|_{L_q(G)} \leqslant C \left(\|f\|_{L_r(G)}^{1-\theta} \left(\sum_{|\alpha|=s} \|D^\alpha f\|_{L_p(G)} \right)^{\theta} + \|f\|_{L_{\tilde{p}}(G)} \right),$$

where $1 \leqslant p, \tilde{p}, q, r < \infty$, $s \in N$, $l \in Z_+$, $l < s$, $\frac{l}{s} < \theta < 1$ provided Gagliardo-Nirenberg equality:

$$l - \frac{n}{q} = \theta \left(s - \frac{n}{p} \right) + (1-\theta) \left(-\frac{n}{r} \right).$$

In 1951 V.P. Il'in established a Gagliardo-Nirenberg type multiplicative inequality for domains satisfying the cone condition in the case $1 \leqslant p, r \leqslant q < \infty$, $l = 0$.

Theorem 2 (G., 2015). Let G be a domain with the flexible σ-cone condition, $1 \leqslant p_j, q, r < \infty$, $p_j < q$, $r \leqslant q$, $p_j > 1$, $s_j \in \mathbb{N}$, $l \in \mathbb{Z}_+$, $l < s_j$, $0 < \theta < 1$ for $j = \overline{1, n}$. Let $r < q$ if $l = 0$, $\sigma = 1$. Then the Gagliardo–Nirenberg type multiplicative inequality

$$\sum_{|\alpha|=l} \|D^\alpha f\|_{L_q(G)} \leqslant C \left(\|f\|_{L_r(G)}^{1-\theta} \left(\sum_{j=1}^{n} \left\| \frac{\partial^{s_j} f}{\partial x_j^{s_j}} \right\|_{L_{p_j}(G)} \right)^{\theta} + \|f\|_{L_r(G)} \right)$$

is valid for all functions f with finite right-hand side provided that the expression

$l - \frac{n}{q} \leqslant \theta \left(s_j - (\sigma - 1) \sum\limits_{i=1, i \neq j}^{n} (s_i - 1) - \frac{\sigma(n-1)+1}{p_j} \right) + (1 - \theta) \left(-\frac{n\sigma}{r} - (\sigma - 1) \left(\sum\limits_{i=1}^{n} s_i - n \right) \right)$ holds for $j = \overline{1, n}$.

We constructed domain with the flexible σ-cone condition, for which inequality from theorem 2 fails for $1 \leqslant p_j, q, r < \infty$, $s_j \in \mathbb{N}$, $l \in \mathbb{Z}_+$, $l < s_j$, $0 < \theta < 1$, $\sigma \geqslant 1$, $\frac{n}{q} < \frac{n}{r} + l$ (for $j = \overline{1, n}$), when expression

$$l - \frac{n}{q} > \theta \left(s_j - (\sigma - 1) \sum_{i=1, i \neq j}^{n} (s_i - 1) - \frac{\sigma(n-1)+1}{p_j} \right) + (1 - \theta) \left(-\frac{n}{r} \right)$$

holds for some $j \in \{1, n\}$.

Thus, for $\sigma = 1$, theorem 2 is sharp.

For proof of results we use averaging from O.V. Besov work "Integral estimates for differentiable functions on irregular domains":

$$\left(D^{\beta}f\right)_t(x) = \int K\left(y, r_{\Gamma}(t), \Gamma(t,x)\right) D^{\beta}f(y)dy$$

where $|\beta| = l$, and averaging kernel
$K\left(\cdot, r_{\Gamma}(t), \Gamma(t,x) - x\right) \in C_0^{\infty}\left(B\left(\Gamma(t,x), r_{\Gamma}(t)\right)\right),$
and also satisfies the certain relations.

One can prove that for almost every $x \in G$

$$\lim_{t\to+0}\left(D^{\beta}f\right)_t(x) = D^{\beta}f(x).$$

From Newton–Leibniz formula it follows that

$$\left|D^{\beta} f(x)\right| \leqslant \int_0^T \left|\frac{\partial}{\partial t}\left(D^{\beta} f\right)_t (x)\right| dt + \left|\left(D^{\beta} f\right)_T (x)\right|$$

for all $T \in (0, T_0]$ (we consider σ-cone with different lengths T).

Estimating terms from right part of the last inequality, we obtain

$$\sum_{|\alpha|=l} \|D^{\alpha} f\|_{L_q(G)} \leqslant C \left(\sum_{j=1}^{n} T^{M_j} \left\| \frac{\partial^{s_j} f}{\partial x_j^{s_j}} \right\|_{L_{p_j}(G)} + T^{M_0} \|f\|_{L_r(G)} \right),$$

The last estimation implies the embedding theorem.

One can obtain the Gagliardo–Nirenberg type multiplicative inequality, using the last estimate and several simply computations.

Let $1 \leqslant m \leqslant n$, $1 \leqslant i_1 < i_2 < \ldots < i_m = n$.
If $\alpha = (\alpha_1, \ldots, \alpha_{i_{j-1}}, \alpha_{i_{j-1}+1}, \ldots, \alpha_{i_j}, \alpha_{i_j+1}, \ldots, \alpha_n)$, then
$\alpha^j = (0, \quad \ldots, 0, \quad \alpha_{i_{j-1}+1}, \ldots, \alpha_{i_j}, \quad 0, \ldots, \quad 0)$.
Later norm of anisotropic Sobolev space will include next sum:

$$\sum_{j=1}^{m} \sum_{\alpha=\alpha^j, |\alpha|=s_j} \|D^\alpha f\|_{L_{p_j}(G)}.$$

If $m = 1$ then it is the sum of norms of all generalized partial derivatives of order s:

$$\sum_{|\alpha|=s} \|D^\alpha f\|_{L_p(G)},$$

If $m = n$ then it is the sum of norms of only unmixed generalized partial derivatives of order s_j:

$$\sum_{i=1}^{n} \left\| \frac{\partial^{s_j} f}{\partial x_i^{s_j}} \right\|_{L_{p_j}(G)}.$$

Theorem 1' (G., 2015). Let G be a domain with the flexible σ-cone condition, $\sigma \geqslant 1$; $s_j, m \in \mathbb{N}$, $l \in \mathbb{Z}_+$, $0 < \theta < 1$, $1 \leqslant m \leqslant n$, $l < s_j$, $1 \leqslant q, r < \infty$, $p_j < q$, $r \leqslant q$, $1 < p_j < \infty$ for $j = \overline{1, m}$. Then the estimate

$$\sum_{|\alpha|=l} \|D^\alpha f\|_{L_q(G)} \leqslant C \left(\sum_{j=1}^{m} \sum_{\alpha=\alpha^j, |\alpha|=s_j} \|D^\alpha f\|_{L_{p_j}(G)} + \|f\|_{L_r(G)} \right) \tag{3}$$

is valid for functions f with finite right-hand side provided that the following expression holds for all $j = \overline{1, m}$:

$$l - \frac{n}{q} \leqslant s_j - (\sigma - 1) \sum_{i=1, i \neq j}^{m} (s_i - 1) - \frac{\sigma(n-1)+1}{p_j}. \tag{4}$$

In isotropic case this theorem coincides with theorem which O.V. Besov proved in 2010.

This theorem is sharp in the class of domains with the flexible σ-cone condition.

Theorem 2' (G., 2015). Let G be a domain with the flexible σ-cone condition, $\sigma \geqslant 1$; $s_j, m \in \mathbb{N}$, $l \in \mathbb{Z}_+$, $0 < \theta < 1$, $1 \leqslant m \leqslant n$, $l < s_j$, $1 \leqslant q, r < \infty$, $p_j < q$, $r \leqslant q$, $1 < p_j < \infty$ for $j = \overline{1, m}$. Let $r < q$ if $l = 0$, $\sigma = 1$. Then the Gagliardo–Nirenberg type multiplicative inequality

$$\sum_{|\alpha|=l} \|D^\alpha f\|_{L_q(G)} \leqslant$$

$$C \left(\|f\|^{1-\theta}_{L_r(G)} \left(\sum_{j=1}^{m} \sum_{\alpha=\alpha^j, |\alpha|=s_j} \|D^\alpha f\|_{L_{p_j}(G)} \right)^\theta + \|f\|_{L_r(G)} \right)$$

is valid for all functions f with finite right-hand side provided that the expression

$$l - \frac{n}{q} \leqslant \theta \left(s_j - (\sigma - 1) \sum_{i=1, i \neq j}^{m} (s_i - 1) - \frac{\sigma(n-1)+1}{p_j} \right) +$$

$$(1-\theta) \left(-\frac{n\sigma}{r} - (\sigma - 1) \left(\sum_{i=1}^{m} s_i - m \right) \right) \quad \text{holds for all} \quad j = \overline{1, m}.$$

If boundary of domain is smooth then theorem 2' coincides with Gagliardo-Nirenberg result.

We constructed domain with the flexible σ-cone condition, for which inequality from theorem 2' fails for $1 \leqslant p_j, q, r < \infty$, $s_j, m \in \mathbb{N}$, $l \in \mathbb{Z}_+$, $l < s_j$, $0 < \theta < 1$, $\sigma \geqslant 1$, $\frac{n}{q} < \frac{n}{r} + l$, $1 \leqslant m \leqslant n$ (for all $j = \overline{1, m}$) when the following expression

$$l - \frac{n}{q} > \theta \left(s_j - (\sigma - 1) \sum_{i=1, i \neq j}^{m} (s_i - 1) - \frac{\sigma(n-1)+1}{p_j} \right) + (1-\theta) \left(-\frac{n}{r} \right)$$

holds for some $j \in \{1, n\}$.

Thus, for $\sigma = 1$ theorem 2' is sharp.

Gagliardo and Nirenberg proved the following miltiplicative inequality without the second term in right part for unbounded domain with smooth boundary

$$\sum_{|\alpha|=l} \|D^\alpha f\|_{L_q(G)} \leqslant C\|f\|_{L_r(G)}^{1-\theta} \left(\sum_{|\alpha|=s} \|D^\alpha f\|_{L_p(G)}\right)^\theta .$$

The question arises whether the multiplicative inequality

$$\sum_{|\alpha|=l} \|D^\alpha f\|_{L_q(G)} \leqslant C\left(\|f\|_{L_r(G)}\right)^{1-\theta} \left(\sum_{j=1}^{m} \sum_{\alpha=\alpha^j, |\alpha|=s_j} \|D^\alpha f\|_{L_{p_j}(G)}\right)^\theta$$

is valid for irregular unbounded domain.

In the case $\sigma > 1$, the answer is negative. Suppose that there exist balls $B(x_R, R)$ of arbitrary radius $R > 0$ that lie in the domain. Let $\xi \in C_0^\infty(B(0,1))$, $\xi \neq 0$. Consider the function $\xi_R = \xi\left(\frac{x-x_R}{R}\right)$. Letting $R \to 0$ и $R \to \infty$, we find that the last inequality can hold only if the following expression is satisfied for $j = \overline{1,m}$

$$l - \frac{n}{q} = \theta\left(s_j - \frac{n}{p_j}\right) + (1-\theta)\left(-\frac{n}{r}\right),$$

but for these values of parametrs multiplicative inequality also fails for some unbounded domains with the flexible σ-cone condition.

For $\sigma = 1$ some generalization is established.

Definition 2. A domain $G \subset \mathbb{R}^n$ is called a domain with unbounded flexible cone condition if, for some $\varkappa > 0$ and any $x \in G$, there exists a piecewise smooth path $\gamma : [0, \infty) \to G, \gamma(0) = x, |\gamma'| \leqslant 1$ almost everywhere, such that $\rho(\gamma(t)) \geqslant \varkappa t$ for all $t > 0$.

Theorem 3 (G., 2015). Let $G \subset \mathbb{R}^n$ be a domain with unbounded flexible cone condition, $1 \leqslant p_j, q, r < \infty$, $s_j, m \in \mathbb{N}$, $l \in \mathbb{Z}_+$, $l < s_j$, $0 < \theta < 1$, $p_j < q$, $r \leqslant q$, $p_j > 1$, $1 \leqslant m \leqslant n$ for $j = \overline{1, m}$. Let $r < q$ if $l = 0$. Then the multiplicative inequality

$$\sum_{|\alpha|=l} \|D^\alpha f\|_{L_q(G)} \leqslant C \left(\|f\|_{L_r(G)}\right)^{1-\theta} \left(\sum_{j=1}^{m} \sum_{\alpha=\alpha^j, |\alpha|=s_j} \|D^\alpha f\|_{L_{p_j}(G)}\right)^\theta$$

is valid for functions f with finite right-hand side proveded that expression

$$l - \frac{n}{q} = \theta \left(s_j - \frac{n}{p_j}\right) + (1-\theta)\left(-\frac{n}{r}\right)$$

is satisfied for all $j = \overline{1, m}$.

A. Yu. Golovko. Additive and Multiplicative Anisotropic Estimates for Integral Norms of Differentiable Functions on Irregular Domains // Proceedings of the Steklov Institute of Mathematics, 2015, Vol. 290, pp. 277–287.

THANK YOU FOR YOUR ATTENTION!