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ESTIMATES FOR THE NORMS OF MONOTONE OPERATORS ON WEIGHTED ORLICZ-LORENTZ CLASSES

1. Ideal quasi-norm (IQN). Ideal Space (IS).
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1. Ideal quasi-norm (IQN). Ideal space (IS).

Let $(\Pi, \Im, \eta)$ be a measure space with nonnegative full σ- finite measure η, $L_0 = L_0(\Pi, \Im, \eta)$ be the space of η- measurable functions $f: \quad \Pi \to R$; $L_0^+ = \{ f \in L_0 : f \ge 0 \}$.

Definition 1.1. *A mapping* $\rho : L_0^+ \to [0, \infty]$ *is an IQN if:*

$$(P1) \quad \rho(f) = 0 \;\Rightarrow\; f = 0; \quad \rho(\alpha f) = \alpha \rho(f), \quad \alpha \ge 0,$$

$$\exists C \in [1, \infty): \quad \rho(f + g) \le C[\rho(f) + \rho(g)];$$

$$(P2) \qquad f \le g \;\Rightarrow\; \rho(f) \le \rho(g)$$; - *monotonicity*

$$(P3) \qquad f_n \uparrow f \;\Rightarrow\; \rho(f_n) \uparrow \rho(f);$$ - *Fatou property*

$$(P4) \qquad \rho(f) < \infty \;\Rightarrow\; f < \infty.$$

Definition 1.2. *Let ρ be an IQN. The IS, generated by ρ is determined as*

$$X = X(\Pi, \Im, \eta) = \left\{ f \in L_0 : \ \| f \|_X = \rho\left(| f | \right) < \infty \right\}. \tag{1.1}$$

Theorem 1.3. *Let X be IS generated by IQN ρ.*

Then, X is quasi-Banach space (Banach space for $C = 1$).

Example. Let $\Pi = R_+ = (0, \infty)$, $\eta = \mu$ be the Lebesgue measure, $L_0 = M$ be the space of μ-measurable functions on R_+. The Lebesgue space: $X = L_p(v)$, $v \in M: \ 0 < v < \infty$,

$$\| f \|_{L_p(v)} = \left(\int_0^\infty | f |^p \, v \, dx \right)^{1/p}, \quad 0 < p < \infty,$$

$$\| f \|_{L_\infty(v)} = ess \sup \left\{ | f(x) v(x) | : \ x \in R_+ \right\}, \quad p = \infty.$$

2. Weighted Orlicz spaces. Some General properties.

Let Θ *be a class of functions* $\Phi:[0,\infty)\rightarrow[0,\infty)$ *such that* $\Phi(0)=0$; Φ *is increasing and left-continuous on* $R_+=(0,\infty)$, $\Phi(t)<\infty,\ t\in R_+$, $\Phi(+\infty)=\infty$.

Always we assume that

$$\Phi\in\Theta;\ v\in M,\ v>0 \text{ almost everywhere on } R_+. \tag{2.1}$$

For $\lambda>0,\ f\in M\equiv M(R_+)$ we denote

$$J_\lambda(f):=\int_0^\infty \Phi\left(\lambda^{-1}|f(x)|\right)v(x)\,dx, \tag{2.2}$$

$$\|f\|_{\Phi,v}=\inf\left\{\lambda>0:\ J_\lambda(f)\le 1\right\}. \tag{2.3}$$

Definition 2.1. *Orlicz space* $L_{\Phi,v}$ *is defined as the set of functions* $f\in M:\ \|f\|_{\Phi,v}<\infty$.

The following result is essentially known (see, for example [2, 11]).

Let $p \in (0,1]$, Φ be p - convex on $[0, \infty)$, that is for $\alpha, \beta \in (0,1]$, $\alpha^p + \beta^p = 1$,

$$\Phi(\alpha t + \beta \tau) \le \alpha^p \Phi(t) + \beta^p \Phi(\tau), \quad t, \tau \in [0, \infty). \tag{2.4}$$

Theorem 2.2. *Let* $\Phi \in \Theta$, $v \in M$, $v > 0$ *, and condition (2.4) be fulfilled. Then,*

1) The triangle inequality takes place in $L_{\Phi, v}$ *: if* $f, g \in L_{\Phi, v}$ *, then* $f + g \in L_{\Phi, v}$ *, and*

$$\| f + g \|_{\Phi, v} \le \left(\| f \|_{\Phi, v}^p + \| g \|_{\Phi, v}^p \right)^{1/p}. \tag{2.5}$$

2) $\| f \|_{\Phi, v}$ *is monotone quasi-norm (norm if* $p = 1$*) :*

$$f \in M, \ | f | \le g \in L_{\Phi, v} \Rightarrow f \in L_{\Phi, v}, \ \| f \|_{\Phi, v} \le \| g \|_{\Phi, v}, \tag{2.6}$$

that has Fatou property: $f_n \in M, \ 0 \le f_n \uparrow f \ \Rightarrow \ \| f \|_{\Phi, v} = \lim_{n \to \infty} \| f_n \|_{\Phi, v}$. (2.7)

Conclusion. *Under conditions of Theorem 2.2* $L_{\Phi, v}$ *forms IS which is quasi – Banach space (Banach space if* $p = 1$*) and has Fatou property (all conditions of theorem 1.3 are fulfilled).*

Example 2.3. Let $v \in M$, $v > 0$; $p \in R_+$, $\Phi(t) = t^p$. Then, Φ is p_1 - convex with $p_1 = \min\{p, 1\}$. We have: $L_{\Phi, v} = L_p(v)$ is Lebesgue space.

Example 2.4. Let $v \in M$, $v > 0$; $\Phi : [0, \infty) \to [0, \infty)$ be *Young function*, that is,

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau, \quad 0 \le \varphi \uparrow \; ; \; \varphi(t-0) = \varphi(t) < \infty, \; t \in R_+. \tag{2.8}$$

Then $\Phi \in \Theta$, and $0 \le \varphi \uparrow \Rightarrow \Phi$ is convex on $[0, \infty)$ (see (2.4) with $p = 1$).

For example
$$\begin{cases} \varphi(0) = 0, \; \varphi(t) = 1, \; t \in (0, 1]; \; \varphi(t) = e^{t-1}, \; t \in (1, \infty) \Rightarrow \\ \Phi(t) = t, \; t \in [0, 1]; \; \Phi(t) = e^{t-1}, \; t \in (1, \infty) \end{cases}$$

Special discretization procedure.

Now we assume that weight – function v satisfies the conditions

$$0 < V(t) := \int_0^t v \, d\tau < \infty, \quad \forall t \in R_+. \tag{2.9}$$

We require that V is strictly increasing and

$$V(+\infty) = \infty. \tag{2.10}$$

For fixed $b > 1$ let us introduce $\{\mu_m\}$ by formulas

$$\mu_m = V^{-1}(b^m) \Leftrightarrow V(\mu_m) = b^m, \quad m \in Z = \{0, \pm 1, \pm 2, \ldots\}, \tag{2.11}$$

where V^{-1} is inverse function for continuous increasing function V. Then

$$0 < \mu_m \uparrow; \quad \lim_{m \to -\infty} \mu_m = 0; \quad \lim_{m \to +\infty} \mu_m = \infty \quad \Rightarrow R_+ = \bigcup_m \Delta_m; \quad \Delta_m = [\mu_m, \mu_{m+1}). \tag{2.12}$$

3. Estimates for the norms of restrictions of monotone operator on the cones in Orlicz space.

Cone Ω of nonnegative decreasing functions:

$$\Omega \equiv \left\{ f \in L_{\Phi,v} : \quad 0 \le f \downarrow \right\}; \tag{3.1}$$

cone $\tilde{\Omega}$ of nonnegative decreasing step-functions:

$$\tilde{\Omega} \equiv \left\{ f \in L_{\Phi,v} : \quad f = \sum_m \alpha_m \chi_{\Delta_m}; \quad 0 \le \alpha_m \downarrow \right\}; \tag{3.2}$$

cone S of nonnegative step-functions

$$S \equiv \left\{ f \in L_{\Phi,v} : \quad f = \sum_m \gamma_m \chi_{\Delta_m}; \quad \gamma_m \ge 0, \ m \in Z \right\}; \tag{3.3}$$

Here, $\Delta_m = [\mu_m, \ \mu_{m+1})$, $V(\mu_m) = b^m$, $m \in Z$, see special discretization (2.9)-(2.12).

Let $(\Pi, \Im, \eta)$ be a measure space with nonnegative full σ- finite measure η; let $L_0 = L_0(\Pi, \Im, \eta)$ be the set of all η- measurable functions $u : \Pi \to R$; $L_0^+ = \{u \in L : u \geq 0\}$. Let $Y = Y(\Pi, \Im, \eta) \subset L_0$ be some IS with a quasi-norm $\|\cdot\|_Y$. Let $P : M^+ \to L_0^+$ be a monotone operator that is

$$f,\ h \in M^+,\quad f \leq h \qquad \mu - almost\ everywhere \quad \Rightarrow \quad P\,f \leq P\,h \qquad \eta - almost\ everywhere .$$

We consider the restrictions of P on the cones in Orlicz space $L_{\Phi,v}$, and determine norms of restrictions

$$\|P\|_{\Omega \to Y} = \sup \left\{ \|P\,f\|_Y \ :\quad f \in \Omega,\ \|f\|_{\Phi,v} \leq 1 \right\}. \tag{3.4}$$

$$\|P\|_{\tilde{\Omega} \to Y} = \sup \left\{ \|P\,f\|_Y \ :\quad f \in \tilde{\Omega},\ \|f\|_{\Phi,v} \leq 1 \right\}.$$

$$\|P\|_{S \to Y} = \sup \left\{ \|P\,f\|_Y \ :\quad f \in S,\ \|f\|_{\Phi,v} \leq 1 \right\}. \tag{3.5}$$

Lemma 3.1. *Let* $\Phi \in \Theta$ *. We assume that* Φ *is* $p-$*convex on* $[0, \infty)$ *with some* $p \in (0,1]$*, and* $v > 0$ *satisfies the conditions (2.9) and (2.10), and realize special discretization (2.11)-(2.12) with fixed* $b > 1$*. Then the following two-sided estimates hold*

$$\| P \|_{\tilde{\Omega} \to Y} \leq \| P \|_{\Omega \to Y} \leq b^{1/p} \| P \|_{\tilde{\Omega} \to Y} ; \tag{3.6}$$

$$\| P \|_{\tilde{\Omega} \to Y} \leq \| P \|_{S \to Y} \leq c(b)^{1/p} \| P \|_{\tilde{\Omega} \to Y} , \tag{3.7}$$

where
$$c(b) = \left[b(b-1)^{-1} \right] > 1.$$

Theorem 3.2. *Let the conditions of Lemma 3.1 be fulfilled***.** *Then the following two-sided estimate holds*

$$c(b)^{-1/p} \| P \|_{S \to Y} \leq \| P \|_{\Omega \to Y} \leq b^{1/p} \| P \|_{S \to Y} , \tag{3.8}$$

Remark 3.3. Theorem 3.2 shows the main goal of the above special discretization. In this theorem we reduce estimates on the cone of *decreasing* functions Ω to the estimates on the cone of *nonnegative* step-functions. In many cases such reduction admits us to apply known results for step-functions or their pure discrete analogues for obtaining needed results on the cone Ω. This approach is realized in Section 4.

4. Associate ideal space for the cone of decreasing functions in the weighted Orlicz space

We apply the results of Section 3 to the important partial case when IS Y coincides with weighted Lebesgue space $L_1\left(R_+ ; g\right), \quad g \in M^+$, and monotone operator $P = I$. In this case

$$\| P \|_{\Omega \to Y} = \sup \left\{ \int_0^\infty f\, g \; d\,t : \quad f \in \Omega; \quad \| f \|_{\Phi,\, v} \le 1 \right\} =: \| g \|'_{\Omega} . \tag{4.1}$$

According to Theorem 3.2 we have

$$\| P \|_{\Omega \to Y} \cong \| P \|_{S \to Y} \ , \tag{4.2}$$

where in our case, by special discretization (2.9)-(2.12), and (3.3),

$$\| P \|_{S \to Y} = \sup \left\{ \sum_{m \in Z} \alpha_m \, g_m : \quad \alpha_m \ge 0; \ \sum_{m \in Z} \Phi\left(\alpha_m \right) \beta_m \le 1 \right\}, \tag{4.3}$$

$$g_m = \int_{\Delta_m} g \, d\,t \ge 0; \quad \beta_m = \int_{\Delta_m} v \, d\,t = b^m \left(b - 1 \right), \quad m \in Z . \tag{4.4}$$

Note that norm (4.3) coincides with the discrete associated Orlicz norm

$$\| \{g_m\} \|_{l'_{\Phi,\beta}} = \sup \left\{ \sum_{m\in Z} \alpha_m |g_m| : \quad \alpha_m \ge 0; \sum_{m\in Z} \Phi(\alpha_m)\beta_m \le 1 \right\}, \tag{4.5}$$

We will describe norm (4.5) explicitly in terms of complementary function Ψ. *For simplicity in our discussion* we restrict ourselves with the case of N - functions:

$$\Phi(s) = \int_0^s \varphi(\tau)d\tau, \quad \varphi \in \Theta \ ; \tag{4.6}$$

Ψ be the complementary function for Φ, that is,

$$\Psi(t) = \int_0^t \psi(\tau)d\tau, \quad t \in [0,\infty]; \quad \psi(\tau) = \inf \{\sigma : \quad \varphi(\sigma) \ge \tau\}, \quad \tau \in [0,\infty]. \tag{4.7}$$

Here, ψ is left inverse function for φ . Then, $\psi \in \Theta$, Ψ is N - function.

Moreover, $\varphi(\sigma)=\inf\{\tau:\ \ \psi(\tau)\geq\sigma\}$, so that Φ itself is the complementary function for Ψ .

Some known formulas:

$$\Psi(t)=\sup_{s\geq 0}[st-\Phi(s)];$$

$$st\leq\Phi(s)+\Psi(t),\ \ s,t\in[0,\infty). \qquad (4.8)$$

Equality in (4.8) takes place if and only if $\varphi(s)=t$ *or* $\psi(t)=s$.

Examples.

1. $\varphi(s)=s^{p-1},\ \ p\in(1,\infty)\ \Rightarrow\ \psi(\tau)=\tau^{p'-1},\ \ 1/p+1/p'=1;$

$$\Phi(s)=s^{p}/p,\ \ \Psi(t)=t^{p'}/p'.$$

2. $\varphi(s)=e^{s}-1\ \Rightarrow\ \psi(\tau)=\ln(1+\tau);$

$$\Phi(s)=e^{s}-s-1,\ \Psi(t)=(t+1)\ln(t+1)-t.$$

Theorem 4.3. *Let* Φ, Ψ *be complementary N- functions; let*

$$0 < V(t) := \int_0^t v\, d\tau < \infty, \quad \forall t \in R_+, \quad V(+\infty) = \infty.$$

For fixed $0 < a < 1$ *the following two-sided estimate holds*

$$\|g\|'_\Omega \cong \|\rho_a(g)\|_{\Psi,v} = \inf\left\{\lambda > 0 : \int_0^\infty \Psi\left(\lambda^{-1}\rho_a(g;t)\right)v(t)dt \le 1\right\}, \tag{4.9}$$

$$\rho_a(g;t) := V(t)^{-1}\int_{\delta_a(t)}^t |g(\tau)|d\tau, \quad \delta_a(t) := V^{-1}(aV(t)),\ t \in R_+. \tag{4.10}$$

Remark 4.4. *Assume additionally that in Theorem 4.3* $\Phi \in (\Delta_2)$*. It means that*

$$\exists C \in (1,\infty): \quad \Phi(2t) \le C\,\Phi(t), \ \forall t \in R_+.$$

Then,

$$\|g\|'_\Omega \cong \left\| V(t)^{-1}\int_0^t |g(\tau)|d\tau \right\|_{\Psi,v}. \tag{4.11}$$

5. Applications to weighted Orlicz – Lorentz classes

For $f \in M$ we introduce distribution function

$$\lambda_f(y) = \mu\{x \in R_+ : |f(x)| > y\}, \ y \in R_+ . \tag{5.1}$$

Let f^* be the *decreasing rearrangement* of function f, that is

$$f^*(t) = \inf\{y \in R_+ : \lambda_f(y) \le t\}, \ t \in R_+ . \tag{5.2}$$

Weighted Orlicz – Lorentz class $\Lambda_{\Phi,v} = \{f \in M(R_+): \ f^* \in L_{\Phi,v}\}$ is equipped by

$$\| f^* \|_{\Phi,v} = \inf\{\lambda > 0 : \ J_\lambda(f^*) \le 1\} . \tag{5.3}$$

$$J_\lambda(f^*) = \int_0^\infty \Phi(\lambda^{-1} f^*(t)) v(t)\, dt, \quad \lambda > 0 , \tag{5.4}$$

We will describe the associated space $\Lambda'_{\Phi,v}$ with the norm

$$\|g\|_*' := \sup\left\{ \int_0^\infty |fg|\,dt : \quad f \in \Lambda_{\Phi,v}\ ;\ \|f^*\|_{\Phi,v} \le 1 \right\}.$$

We use the following properties of decreasing rearrangements

$$0 \le h \downarrow \ \Rightarrow\ \sup\left\{ \int_0^\infty |fg|\,dt : \quad f \in M,\ f^* = h \right\} = \int_0^\infty h g^*\,dt;$$

$$h \in \Omega \ \Leftrightarrow\ \exists f \in \Lambda_{\Phi,v} :\ f^* = h\ .$$

Then,

$$\|g\|_*' = \sup\left\{ \int_0^\infty h\ g^*\,dt : \quad h \in \Omega;\ \|h\|_{\Phi,v} \le 1 \right\} = \|g^*\|_\Omega' .$$

To estimate $\|g^*\|_\Omega'$ we apply Theorem 4.3 and obtain the following result.

Theorem 5.1. *Let* Φ, Ψ *be complementary Young functions; let*

$$0 < V(t) := \int_0^t v\, d\tau < \infty, \quad \forall t \in R_+, \quad V(+\infty) = \infty.$$

For fixed $0 < a < 1$ *the following two-sided estimate holds*

$$\| g \|_*^{/} \cong \left\| \rho_a\left(g^*\right) \right\|_{\Psi, v} = \inf\left\{ \lambda > 0 : \int_0^\infty \Psi\left(\lambda^{-1} \rho_a\left(g^*; t\right)\right) v(t) dt \le 1 \right\}, \tag{5.5}$$

$$\rho_a\left(g^*; t\right) := V(t)^{-1} \int_{\delta_a(t)}^t g^*(\tau)\, d\tau, \qquad \delta_a(t) := V^{-1}\left(aV(t)\right),\ t \in R_+.$$

Remark 5.2. *Assume additionally that in Theorem 5.1* $\Phi \in \left(\Delta_2\right)$*.Then,*

$$\| g \|_*^{/} \cong \left\| V(t)^{-1} \int_0^t g^*(\tau)\, d\tau \right\|_{\Psi, v}. \tag{5.6}$$

Let $\left(\Pi, \Im, \eta\right)$ be the measure space with nonnegative full σ- finite measure η; let $L_0 = L_0\left(\Pi, \Im, \eta\right)$ be the set of all η- measurable functions $u : \Pi \to R$; $L_0^+ = \left\{u \in L :\ u \ge 0\right\}$.

Theorem 5.3. *Let* $Y \subset L_0$ *be some IS with a quasi-norm* $\left\|\cdot\right\|_Y$, $P :\ M^+ \to L_0^+$ *be a monotone operator satisfying the following condition: there exists* $C \in \left[1, \infty\right)$ *such that*

$$\left\| P\, f \right\|_Y \le C \left\| P\, f^* \right\|_Y , \quad f \in M^+\left(R_+\right) . \tag{5.7}$$

Then,

$$\left\| P \right\|_{\Omega \to Y} \le \left\| P \right\|_{\Lambda^+_{\Phi, v} \to Y} \le C \left\| P \right\|_{\Omega \to Y} .$$

Corollary 5.4. *Under assumptions of Theorem 5.3 we have*

$$\left\| P \right\|_{\Lambda^+_{\Phi, v} \to Y} \cong \left\| P \right\|_{S \to Y} .$$

Remark 5.5. In Theorem 5.3 we reduce estimates for monotone operator P on the weighted Orlicz – Lorentz class $\Lambda_{\Phi,v}$ to the estimates on the cone S of nonnegative step-functions from Orlicz space $L_{\Phi,v}$. In many cases such reduction admits us to apply known results for step-functions or their pure discrete analogues for obtaining needed results on $\Lambda_{\Phi,v}$.

Remark 5.6. Theorem 5.3 covers the operators

$$(Pf)(x)=\int_0^\infty k(x,\tau)f(\tau)\,d\tau, \quad x\in\Pi, \quad f\in M^+ \tag{5.8}$$

with nonnegative measurable k on $\Pi\times R_+$, such that $k(x,\tau)$ is decreasing and right-continuous as function of $\tau\in R_+$. Indeed, for η- almost all $x\in\Pi$ we have by Hardy's lemma,

$$(Pf)(x)\le\int_0^\infty k(x,\tau)f^*(\tau)\,d\tau=(Pf^*)(x).$$

Then, for IS $Y=Y(\Pi)$ we have $\|Pf\|_Y\le\|Pf^*\|_Y$.

Remark 5.7. Theorem 5.3 covers maximal operator $\mathrm{M}: M_+(R_+) \to M_+(R_+)$,

$$(\mathrm{M} f)(x) = \sup\left\{ |\Delta|^{-1} \int_{\Delta} f(\tau)\, d\tau : \quad \Delta \subset R_+;\ x \in \Delta \right\},$$

when $Y = Y(R_+)$ is an RIS. Indeed, for any RIS Y there exists unique RIS $\tilde{Y} = \tilde{Y}(R_+)$:

$\| g \|_Y = \| g^* \|_{\tilde{Y}}$, $\quad g \in M(R_+)$, see [11; Ch. 2]. Let us note that $(\mathrm{M} f^*)^* = \mathrm{M} f^*$. Then, we have

$$\| \mathrm{M} f \|_Y = \| (\mathrm{M} f)^* \|_{\tilde{Y}}, \quad \| \mathrm{M} f^* \|_Y = \| \mathrm{M} f^* \|_{\tilde{Y}} .$$

It is well-known that $C \in [1, \infty)$: $(\mathrm{M} f)^*(x) \le C (\mathrm{M} f^*)(x)$, see [11;Ch.2] .Therefore,

$$\| \mathrm{M} f \|_Y = \| (\mathrm{M} f)^* \|_{\tilde{Y}} \le C \| \mathrm{M} f^* \|_{\tilde{Y}} = C \| \mathrm{M} f^* \|_Y$$

This coincides with (5.7) for $P = \mathrm{M}$, and Theorem 5.3 is applicable here.

Thanks for your attention!

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