Some s-numbers of an integral operator of Hardy type in Banach function spaces

Amiran Gogatishvili

Institute of Mathematics
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The Hardy-type operator

$$Tf(x) = T_{a,I,u,v}f(x) = v(x)\int_a^x f(t)u(t)dt,$$

where $I = (a, b)$, u and v are given real valued nonnegative functions on I with $|\{x : u(x) = 0\}| = |\{x : v(x) = 0\}| = 0$.
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This operator appears naturally in the theory of differential equations and it is important to establish when operators of this kind have properties such as boundedness, compactness, and to estimate their eigenvalues, or their approximation numbers.

Let $L(I)$ be the space of all Lebesgue-measurable real functions on $I = (a, b)$, where $-\infty < a < b < +\infty$. A Banach subspace E of $L(I)$ is said to be a Banach function space (BFS) if:

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Given a Banach function space E, its associate space E' consists of those $g \in S$ such that $f \cdot g \in L^1$ for every $f \in E$ with norm

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E' is a BFS on I and a closed norm fundamental subspace of the conjugate space E^*.

We say that the space E has absolutely continuous norm (AC-norm) if for all

$$f \in E, \quad \|f \chi_{X_n}\|_E \to 0$$

for every sequence of measurable sets $\{X_n\} \subset I$ such that $\chi_{X_n} \to 0$ a.e.

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We recall that E is said to be strictly convex if whenever $x, y \in E$ are such that $x \neq y$ and $\|x\| = \|y\| = 1$, and $\lambda \in (0, 1)$, then $\|\lambda x + (1 - \lambda)y\| < 1$.
This simply means that the unit sphere in E does not contain any line segment.

By Π we denote the family of all sequences $\mathcal{Q} = \{I_i\}$ of disjoint intervals in I such that $I = \cup_{I_i \in \mathcal{Q}} I_i$.
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Everywhere in the sequel by $l_{\mathcal{Q}},\ (\mathcal{Q} \in \Pi)$ we denote a Banach sequence space (BSS) (indexed by a partition $\mathcal{Q} = \{I_i\}$ of I), meaning that axioms 1)-4) are satisfied with respect to the counting measure.

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Let J be an arbitrary interval of I. By $E(J)$ we denote the "restriction" of the space E to J; $E(J) = \{f\chi_J : \ f \in E\}$, with the norm $\|f\|_{E(J)} = \|f\chi_J\|_E$.

Definition

Let $I = \{I_{\mathcal{Q}}\}_{\mathcal{Q}\in\Pi}$ be a family of BSSs. A BFS E is said to satisfy a uniform upper (lower) $I-$estimate if there exists a constant $C > 0$ such that for every $f \in E$ and $\mathcal{Q} \in \Pi$ we have

$$\|f\|_E \leq C\|\sum_{I_i\in\mathcal{Q}} \|f\chi_{I_i}\|_E \cdot e_{I_i}\|_{I_{\mathcal{Q}}} \quad \left(\|\sum_{I_i\in\mathcal{Q}} \|f\chi_{I_i}\|_E \cdot e_{I_i}\|_{I_{\mathcal{Q}}} \leq C\|f\|_E\right).$$

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Note also that if E simultaneously satisfies upper and lower $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\Pi}$ estimates then E' simultaneously satisfies upper and lower $l' = \{l'_{\mathcal{Q}}\}_{\mathcal{Q}\in\Pi}$ estimates.

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This Definition was introduced by Kopaliani. The idea behind it is simply to generalize the following property of the Lebesgue norm:

$$\|f\|_{L^p}^p = \sum_i \|f\chi_{\Omega_i}\|_{L^p}^p$$

for a partition of $\mathbb{R}^n$ into measurable sets Ω_i.

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The notions of uniform upper (lower) l−estimates, when $l_{\mathcal{Q}_1} = l_{\mathcal{Q}_2}$ for all $\mathcal{Q}_1, \mathcal{Q}_2 \in \Pi$, were introduced by Berezhnoi .

Theorem

Let E be a BFS. Then the following assertions are equivalent:

1) *There is a family $I = \{I_\mathcal{Q}\}_{\mathcal{Q}\in\Pi}$ of BSSs such that E satisfies simultaneously upper and lower $I = \{I_\mathcal{Q}\}_{\mathcal{Q}\in\Pi}$ estimates.*

2) *There exists a constant $C > 0$ such that for any $f \in E$ and $\mathcal{Q} \in \Pi$,*

$$\frac{1}{C}\|f\|_E \leq \left\| \sum_{I_i\in\mathcal{Q}} \frac{\|f\chi_{I_i}\|_E}{\|\chi_{I_i}\|_E} \cdot \chi_{I_i} \right\|_E \leq C\|f\|_E. \tag{1}$$

3) *There exists a constant $C_1 > 0$ such that*

$$\sum_{I_i\in\mathcal{Q}} \|f\chi_{I_i}\|_E \|g\chi_{I_i}\|_{E'} \leq C_1\|f\|_E\|g\|_{E'} \tag{2}$$

for any $\mathcal{Q} \in \Pi$ and every $f \in E$, $g \in E'$.

Theorem

Let E and F be BFSs with the following property: there exists a family of BSS $I = \{I_Q\}_{Q \in \Pi}$ such that E satisfies a uniform lower I-estimate and F a uniform upper I-estimate. Suppose that $u\chi_{(a,x)} \in E'$ and $v\chi_{(x,b)} \in E$ whenever $a < x < b$.. Then T is a bounded operator from E into F if and only if

$$\sup_{a<t<b} A(t) = \sup_{a<t<b} \|v\chi_{(t,b)}\|_F \|u\chi_{(a,t)}\|_{E'} < \infty.$$

We observe that similar results hold when we replace v and u by $v\chi_J$ and $u\chi_J$ respectively, where J is any subinterval of I.

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Theorem

Let $J = (c, d)$ be any interval of I; let E and F be BFS for which there exists a family of BSS $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\Pi}$ such that E satisfies a uniform lower l-estimate and F a uniform upper l-estimate. Then the operator

$$T_J f(x) = v(x)\chi_J(x)\int_a^x u(t)\chi_J(t)f(t)dt$$

is bounded from E into F if and only if

$$A_J = \sup_{t\in J} A_J(t) = \sup_{t\in J} \|v\chi_J\chi_{(t,d)}\|_F \|u\chi_J\chi_{(c,t)}\|_{E'} < \infty.$$

Moreover $A_J \le \|T_J\| \le K \cdot A_J$, where $K \ge 1$ is a constant independent of J.

Theorem

Let $T : E \longrightarrow F$ *be bounded, where* E, F *are BFS with AC- norms. Then* T *is compact from* E *to* F *if and only if the following two statements are satisfied:*

$$\lim_{x \longrightarrow a+} \sup_{a<r<x} \|v\chi_{(r,x)}\|_F \|u\chi_{(a,r)}\|_{E'} = 0,$$

and

$$\lim_{x \longrightarrow b-} \sup_{x<r<b} \|v\chi_{(r,b)}\|_F \|u\chi_{(x,r)}\|_{E'} = 0.$$

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Note that if E and F have AC-norms and $u \in E'$, $v \in F$ then $T : E \to F$ is compact.

$B(E, F)$ will denote the space of all bounded linear maps of E to F.
Given a closed linear subspace M of E, the embedding map of M into E will be denoted by J_M^E and the canonical map of E onto the quotient space E/M by Q_M^E. Let $S \in B(E, E)$. Then the modulus of injectivity of T is

$$j(S) = \sup\{\rho \geq 0 : \ \|Sx\|_E \geq \rho \|x\|_E \ \text{ for all } \quad x \in E\}.$$

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Definition

Let $S \in B(E, E)$ and $n \in \mathbb{N}$. Then the nth approximation, isomorphism, Gelfand, Bernstein and Kolmogorov numbers of S are defined by

$$a_n(S) = \inf\{\|S - P\| : \ P \in B(E, E), \ \operatorname{rank}(P) < n\};$$

$$i_n(S) = \sup\{\|A\|^{-1}\|B\|^{-1}\},$$

where the supremum is taken over all possible Banach spaces G with $\dim G \geq n$ and maps $A \in B(E, G), \ B \in B(G, E)$ such that ASB is the identity on G;

$$c_n(S) = \inf\{\|SJ_M^E\| : \ \operatorname{codim}(M) < n\};$$

$$b_n(S) = \sup\{j(SJ_M^E); \ \dim(M) \geq n\};$$

$$d_n(S) = \inf\{\|Q_M^E S\| : \ \dim(M) < n\}.$$

Below $s_n(S)$ denotes any of the nth approximation, isomorphism, Gelfand, Kolmogorov or Bernstein numbers of the operator S.
Some of the facts concerning the numbers $s_n(S)$ are summarized in the following theorem

Theorem

Let $S \in B(E, E)$ *and* $n \in \mathbb{N}$. *Then*

$$a_n(S) \geq c_n(S) \geq b_n(S) \geq i_n(S)$$

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Theorem

Suppose that $1 < p < \infty$, $v \in L^p(a,b)$, $u \in L^q(a,b)$ *where* $1/p + 1/q = 1$. *Then for* $T: L^p(a,b) \to L^p(a,b)$ *we have*

$$\lim_{n\to\infty} ns_n(T) = \frac{1}{2}\gamma_p \int_a^b u(x)v(x)dx,$$

where $\gamma_p = \pi^{-1} p^{1/q} q^{1/p} \sin(\pi/p)$.

We say that a space E fulfills the Muckenhoupt condition if for some constant $C > 0$ and for all intervals $J \subset I$ we have

$$\frac{1}{|J|} \|\chi_J\|_E \|\chi_J\|_{E'} \leq C.$$

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Note that if E fulfills the Muckenhoupt condition, then using Hölders inequality we obtain

$$\frac{1}{|J|}\int_J |f(x)|dx \leq C\frac{\|f\chi_J\|_E}{\|\chi_J\|_E},$$

and if additionally E simultaneously satisfies upper and lower $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\Pi}$ estimates, then from (1) we obtain

$$\left\|\sum_{I_i\in\mathcal{Q}}\chi_{I_i}\frac{1}{|I_i|}\int_{I_i}|f(x)|dx\right\|_E \leq C_1\|f\|_E,$$

where $C_1 > 0$ is an absolute constant independent of the partition $\mathcal{Q}$ of I.

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If for a space E we have the Muckenhoupt condition and (1), we denote this by writing $E \in \mathcal{M}$.

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Note that in the case of a reflexive variable exponent Lebesgue space the condition $L^{p(\cdot)} \in \mathcal{M}$ implies the boundedness of the Hardy-Littlewood maximal operator in $L^{p(\cdot)}$.

The main result

Theorem

Let E be BFS belong to the class $\mathcal{M}$. Let the spaces E, E^* be strictly convex and assume that E and E' have AC- norms. Suppose $u \in E'$, $v \in E$. Then there exists constants $C_1 = C_1(E), C_2 = C_2(E) > 0$ such that, for the map $T : E \to E$

$$C_1 \int_a^b u(x)v(x)dx \le \liminf_{n\to\infty} ns_n(T) \le \limsup_{n\to\infty} ns_n(T) \le C_2 \int_a^b u(x)v(x)dx.$$

Given a measurable function $p(\cdot) : (a, b) \to [1, +\infty)$,
$L^{p(\cdot)}(a, b)$ denotes the set of measurable functions f on (a, b) such that for some $\lambda > 0$,

$$\int_{(a,b)} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

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We say that a function $p : (a,b) \to (1, \infty)$ is log-Hölder continuous if there exists $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{\log(e + 1/|x-y|)} \quad \text{for all} \quad x, y \in (a,b) \quad \text{and} \quad x \neq y.$$

Denote by $\mathcal{P}_{\log}$ the set of all log-Hölder continuous exponents that satisfy

$$p_- = \text{ess inf}_{x\in(a,b)} p(x) > 1, \quad p_+ = \text{ess sup}_{x\in(a,b)} p(x) < \infty.$$

Proposition

Let $p(\cdot) \in \mathcal{P}_{\log}$. *Then* $L^{p(\cdot)}(a, b) \in \mathcal{M}$.

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Note that there exist another class of exponents giving rise to property (1). Indeed, let $p(\cdot) : [0,1] \to [1,+\infty)$ be log-Hölder continuous, $w(t) = \int_a^t l(u)du$, $t \in (a,b)$, $w(b) = 1$, $l(u) > 0$ $(u \in (a,b))$. Then $L^{p((w(\cdot))}(a,b)$ has property (1)).

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Corollary

Let $p(\cdot) \in \mathcal{P}_{\log}$ *and*
$v \in L^{p(\cdot)}(a,b)$, $u \in L^{q(\cdot)}(a,b)$ $(1/p(x) + 1/q(x) = 1,\ x \in (a,b))$. *Then* T *acts from the variable exponent space* $L^{p(\cdot)}(a,b)$ *to itself and*

$$C_1 \int_{(a,b)} u(x)v(x)dx \le \liminf_{n\to\infty} ns_n(T) \le \limsup_{n\to\infty} ns_n(T) \le C_2 \int_{(a,b)} u(x)v(x)dx.$$

We say that an exponent $p(\cdot) \in \mathcal{P}_{\log}$ is strongly log-Hölder continuous (and write $p(\cdot) \in \mathcal{SP}_{\log}$) if there is an increasing continuous function defined on $[0, b-a]$ such that $\lim_{t \to 0+} \psi(t) = 0$ and

$$-|p(x)-p(y)| \ln |x-y| \leq \psi(|x-y|) \quad \text{for all } x, y \in (a, b) \text{ with } 0 < |x-y| < 1/2.$$

Theorem

Let $p(\cdot) \in \mathcal{SP}_{\log}$ and $u = v = 1$. Then T acts from the variable exponent space $L^{p(\cdot)}(a, b)$ to itself and

$$\lim_{n \to \infty} n s_n'(T) = \frac{1}{2\pi} \int_I (q(x)p(x)^{p(x)-1})^{1/p(x)} \sin(\pi/p(x)) dx,$$

where $s_n'(T)$ stands for any of the n-th approximation, Gelfand, Kolmogorov and Bernstein numbers of T.

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Thank you for atention!