Osnabrück University, Germany • Lars Diening

Nonlinear Calderón-Zygmund theory

joint work with . . .

Breit

Cianchi

Kuusi

Schwarzacher

Nonlinear Calderón-Zygmund theory ???

Let us start with linear Calderón-Zygmund theory!

Consider the linear PDE on $\mathbb{R}^n$ with $f \in L^q$

$$-\Delta u = -\sum_j \frac{\partial^2}{\partial x_j^2} u = f.$$

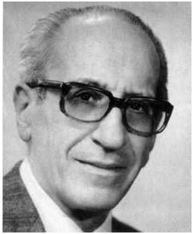
Then $u = G * f$ with $G(x) = c_n |x|^{2-n}$.

Theorem (singular integral operator)

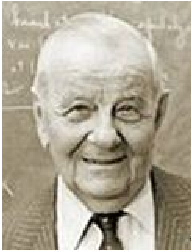
Then $f \mapsto \nabla^2 u$ is bounded from L^q to L^q for $1 < q < \infty$.

Idea: $\nabla^2 u = (\nabla^2 G) * f$ with $|\nabla^2 G(x)| \sim |x|^{-n}$.

and cancellation properties of $\nabla^2 G$



Calderón



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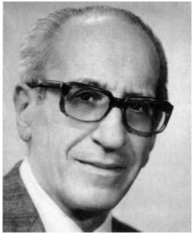

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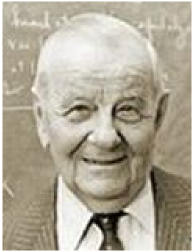

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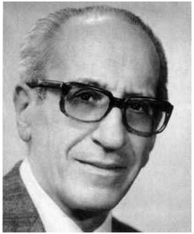

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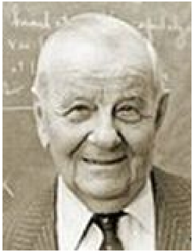

Zygmund

Data in divergence form

Consider now the same problem with $f = \operatorname{div} F$ and $F \in L^q$

$$-\operatorname{div}(\nabla u) = -\Delta u = \operatorname{div} F.$$

Again $F \mapsto \nabla u$ is a *singular integral operator*.

Theorem

The mapping $F \mapsto \nabla u$ is bounded from L^q to L^q for $1 < q < \infty$.

In terms of regularity we can cancel the divergence:

$$\cancel{-\operatorname{div}}(\nabla u) = \cancel{\operatorname{div}} F$$

Question: What about $q = \infty$?

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The limiting case L^∞

Consider $-\operatorname{div}(\nabla u) = \operatorname{div} F$. Then unfortunately $\|\nabla u\|_\infty \not\lesssim \|F\|_\infty$.

Counterexample: Let $u := x_1 \ln |x|$, then $\nabla u \notin L^\infty$ but

$$-\Delta u = \operatorname{div}\left(|x|^{-2} \begin{pmatrix} 2x_1x_2 \\ x_1^2 - x_2^2 \end{pmatrix}\right) \in \operatorname{div}(L^\infty).$$

Singular integrals are not bounded on L^∞ but on BMO.

BMO – space of bounded mean oscillation

$$w \in \mathrm{BMO} \quad :\Leftrightarrow \quad \|w\|_{\mathrm{BMO}} = \sup_{B \text{ is a ball}} \fint_B |w - \langle w\rangle_B|\,dx < \infty.$$

Example: $\ln|x| \in \mathrm{BMO} \setminus L^\infty$

Theorem

$F \in \mathrm{BMO}$ implies $\nabla u \in \mathrm{BMO}$.

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The space BMO

BMO – space of bounded mean oscillation

$$w \in \mathrm{BMO} \quad :\Leftrightarrow \quad \|w\|_{\mathrm{BMO},\Omega} = \sup_{B \text{ is a ball}} \fint_B |w - \langle w \rangle_B| \, dx < \infty.$$

This is the correct substitute for L^∞!

We have $L^\infty \hookrightarrow \mathrm{BMO} \hookrightarrow L^{\exp}_{\mathrm{loc}}$.

Recall $\ln|x| \in L^\infty \setminus \mathrm{BMO}$.

Define the maximal operator $(M^\sharp F)(x) := \sup_{B \ni x} \fint_B |F - \langle F \rangle_B| \, dx$.

Then: $\quad F \in \mathrm{BMO} \;\Leftrightarrow\; M^\sharp F \in L^\infty$.

Finally nonlinear!

Laplacian:

The solutions $u : \mathbb{R}^n \to \mathbb{R}^N$ of

$$-\mathrm{div}(\nabla u) = \mathrm{div} F$$

minimize the energy $\mathcal{J}(w) := \int \frac{1}{2}|\nabla w|^2\,dx + \int \nabla w \cdot F\,dx.$

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p-Laplacian: (with $1 < p < \infty$)

Minimizers of $\mathcal{J}(w) := \int \frac{1}{p}|\nabla w|^p\,dx + \int \nabla w \cdot F\,dx$ satisfy

$$-\mathrm{div}(A(\nabla u)) := -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mathrm{div} F.$$

In terms of regularity we want to compare F with $A(\nabla u)$:

$$-\cancel{\mathrm{div}}(A(\nabla u)) = \cancel{\mathrm{div}} F.$$

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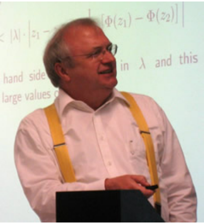
Foundation of nonlinear Calderón-Zygmund theory!

$$-\mathrm{div}(A(\nabla u)) = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mathrm{div} F.$$

Weak solutions

$F \in L^{p'}$ implies $\nabla u \in L^p$ and $A(\nabla u) \in L^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$.



Iwaniec

Theorem (Iwaniec '82, DiBenedetto, Manfredi '93)

$F \in L^q$ implies $A(\nabla u) \in L^q$ for all $q \in [p', \infty)$.

Idea: Locally compare with p-harmonic functions, i.e. $F = 0$.

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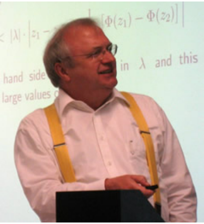

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Harmonic vs. p-harmonic

Linear:

If h is harmonic, i.e. $-\Delta h = 0$, then $h \in C^\infty$.

Non-linear:

If h is p-harmonic, i.e. $-\mathrm{div}(A(\nabla h)) = -\mathrm{div}(|\nabla h|^{p-2}\nabla h) = 0$,

then $\nabla h \in C^{0,\alpha}$ and $A(\nabla h) \in C^{0,\beta}$.

[Uraltseva; Uhlenbeck; Acerbi-Fusco; Tolksdorf; ...]

Decay estimate [e.g. Diening, Stroffolini, Verde '09]:

Let $V := |\nabla h|^{\frac{p}{2}} \frac{\nabla h}{|\nabla h|}$, then for $0 < r < R$

$$\fint_{B_r} |V - \langle V \rangle_{B_r}|^2 \, dx \lesssim \Big(\frac{r}{R}\Big)^{2\gamma} \fint_{B_R} |V - \langle V \rangle_{B_R}|^2 \, dx.$$

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Bad $p-$harmonic functions in the plane (2D!)

Construction by Dobrowolski

Find positive harmonic function $h \in W_0^{1,p}(\Omega)$ on $\Omega := (0,\infty)^2$ and reflect this to other quadrants. Then h is p-harmonic and $h \in C^\alpha(\mathbb{R}^n)$ with $\alpha = \frac{7p-6+\sqrt{p^2+12p-12}}{6p-6}$.

$D := \nabla u$
$A := A(\nabla u)$
$V := |\nabla u|^{\frac{p}{2}} \frac{\nabla u}{|\nabla u|}$

Optimal regularity in 2D

This regularity is the best possible.
[Aaronson '88],
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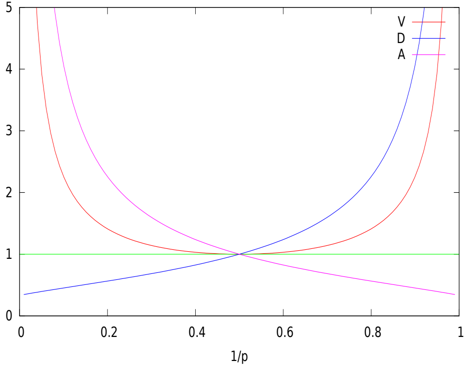

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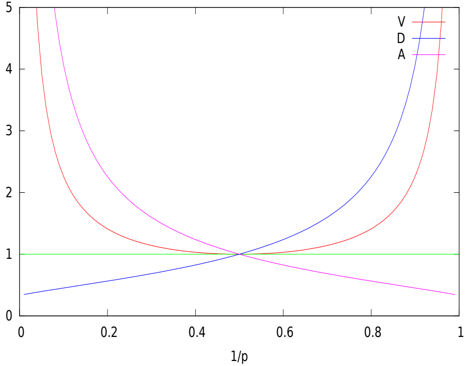

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The BMO case

Let $-\mathrm{div}(A(\nabla u)) = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mathrm{div}(F)$.

Theorem (Manfredi, Di Benedetto 93')

If $p \geq 2$, then

$$\|\nabla u\|_{\mathrm{BMO}(B)}^{p-1} \leq c\,\|F\|_{\mathrm{BMO}(2B)} + c\left(\fint_{2B}\left|\frac{u-\langle u\rangle_{2B}}{r_B}\right|^p dx\right)^{\frac{p-1}{p}}.$$

Theorem (Diening, Kaplický, Schwarzacher '12)

For any $p > 1$,

$$\|A(\nabla u)\|_{\mathrm{BMO}(B)} \leq c\,\|F\|_{\mathrm{BMO}(2B)} + c\fint_{2B}|A(\nabla u) - \langle A(\nabla u)\rangle_{2B}|\,dx.$$

Main step: Compare locally to p-harmonic functions. ($\rightarrow$ next slide)

Idea: Comparison with p-harmonic functions:

Let $-\mathrm{div}(A(\nabla u)) = \mathrm{div} F$. Recall $|V(Q)|^2 = A(Q) \cdot Q = |Q|^p$.

Let h be p-harmonic on B with $h = u$ on ∂B. Then

$$\underbrace{\big\langle A(\nabla u) - A(\nabla h), \nabla u - \nabla h \big\rangle}_{\eqsim \|V(\nabla u) - V(\nabla h)\|_2^2} = \big\langle F - \langle F \rangle_B, \nabla u - \nabla h \big\rangle.$$

Comparison helps to transfer decay estimate of $V(\nabla h)$ to $V(\nabla u)$.

Problem: Different growth of $|A|$ and $|V|^2$, namely $p-1$ vs. p.

Tools: John-Nirenberg for BMO, reverse Hölder estimate.

Requires $F \in \mathrm{BMO}$!

New pointwise estimate!

Based on similar but refined techniques we get:

Theorem [Breit, Cianchi, Diening, Kuusi, Schwarzacher '15]

Global version

$$M^\sharp(A(\nabla u)) \lesssim M^\sharp_{p'}(F).$$

Local version on ball B

$$M^\sharp(A(\nabla u)) \lesssim M^\sharp_{p'}(F) + \left(\fint_{2B} |A(\nabla u) - \langle A(\nabla u)\rangle_{2B}|^{p'}\, dx \right)^{\frac{1}{p'}}$$

Maximal operator: $(M^\sharp_{p'}F)(x) = \sup_{B \ni x} \left(\fint_B |F - \langle F\rangle_B|^{p'}\, dx \right)^{\frac{1}{p'}}$.

Can replace $M^\sharp(A(\nabla u))$ by $M^\sharp_{\min\{2,p'\}}(A(\nabla u))$

How does this help?

Our pointwise estimate allows to reprove all previous results!

$$M^\sharp(A(\nabla u)) \lesssim M^\sharp_{p'}(F)$$

Case L^q: For $q > p'$

$$\|A(\nabla u)\|_q \lesssim \|M^\sharp(A(\nabla u))\|_q \lesssim \|M^\sharp_{p'}(F)\|_q \lesssim \|F\|_q$$

Case BMO:

$$\|A(\nabla u)\|_{\mathrm{BMO}} = \|M^\sharp(A(\nabla u))\|_\infty \lesssim \|M^\sharp_{p'}(F)\|_\infty \lesssim \|F\|_{\mathrm{BMO}}.$$

More examples:

Estimates in Lorentz spaces $L^{q,s}$ with $q > p'$ and $q \in [1, \infty]$ follow as easily. Also many sharper endpoint estimates follow.

Beyond BMO (Campanato spaces)

Let us use the characterization of $C^{0,\alpha}$ by mean oscillations:

Maximal operator: $(M^\sharp_{p',\omega}F)(x) = \sup_{B_r \ni x} \frac{1}{\omega(r)} \Big(\fint_{B_r} |F - \langle F\rangle_{B_r}|^{p'}\,dx\Big)^{\frac{1}{p'}}.$

Theorem [Breit, Cianchi, Diening, Kuusi, Schwarzacher '15]

Global version

$$M^\sharp_\omega(A(\nabla u)) \lesssim M^\sharp_{p',\omega}(F).$$

This implies for example with $\omega(t) = t^\beta$

$$\|A(\nabla u)\|_{C^{0,\beta}} \lesssim \|F\|_{C^{0,\beta}}$$

up to the regularity of p-harmonic functions.

Potential type estimates

For $-\mathrm{div}(A(\nabla u)) = \mathrm{div}(F)$ we get the potential type estimate:

Theorem [Breit, Cianchi, Diening, Kuusi, Schwarzacher '15]

$$|A(\nabla u(x))| \lesssim \int_0^\infty \bigg(\fint_{B_r(x)} \bigg| \frac{|F - \langle F\rangle_{B_r(x)}|}{r} \bigg|^{p'} dy \bigg)^{\frac{1}{p'}} \frac{dr}{r}.$$

(Unconventional Havin-Maz'ya-Wolff type potential).

Mingione

Compare this with the case $-\mathrm{div}(A(\nabla u)) = f$.

Theorem (Mingione, Kuusi)

$$|A(\nabla u(x))| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}} \, dy.$$

They have more wonderful results!

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Finally!

Some words to the proof . . .

Natural quantities

Recall: $$A(Q) = |Q|^{p-1}\frac{Q}{|Q|}, \qquad V(Q) = |Q|^{\frac{p}{2}}\frac{Q}{|Q|}.$$

Define

$$\varphi_{p,|Q|}(t) := (|Q| + t)^{p-2}\, t^2 \qquad \text{shifted N-function,}$$

$$\varphi_{p',|A(Q)|}(t) := (|A(Q)| + t)^{p'-2}\, t^2 \qquad \text{its complementary N-function.}$$

Natural quantities

$$\begin{aligned} \langle A(P) - A(Q), P - Q\rangle &\eqsim |V(P) - V(Q)|^2 \\ &\eqsim \varphi_{p,|Q|}(|P-Q|) \\ &\eqsim \varphi_{p',|A(Q)|}(|A(P) - A(Q)|). \end{aligned}$$

Reverse Hölder estimate

The standard test function $(u-q)\eta^s$ gives

$$\fint_B |V(\nabla u) - V(Q)|^2 dx \lesssim \bigg(\fint_{2B} |V(\nabla u) - V(Q)|^{2\sigma} dx\bigg)^{\frac{1}{\sigma}} + \fint_{2B} \varphi_{p',|A(Q)|}(|F - F_0|) dx.$$

In other words

$$\fint_B \varphi_{p',|A(Q)|}(|A(\nabla u) - A(Q)|)\, dx \lesssim \bigg(\fint_{2B} (\varphi_{p',|A(Q)|}(|A(\nabla u) - A(Q)|))^{\sigma} dx\bigg)^{\frac{1}{\sigma}} + \cdots .$$

If we can reduce it a little we can reduce it the full range:

$$\fint_B \varphi_{p',|A(Q)|}(|A(\nabla u) - A(Q)|)\, dx \lesssim \varphi_{p',|A(Q)|}\bigg(\fint_{2B} |A(\nabla u) - A(Q)|\, dx\bigg) + \cdots .$$

Decay estimates of p-harmonic function

Later we compare u locally with a p-harmonic function h.

We need [Diening, Stroffolini, Verde '09]

$$\sup_{x,y\in\theta B} \big|V(\nabla h)(x) - V(\nabla h)(y)\big|^2\,dx \lesssim \theta^{2\alpha} \fint_B \big|V(\nabla h)(x) - \langle V(\nabla h)\rangle_B\big|^2\,dx$$

and [Diening, Kaplický, Schwarzacher '12]

$$\sup_{x,y\in\theta B} \big|A(\nabla h)(x) - A(\nabla h)(y)\big|\,dx \lesssim \theta^{\beta} \fint_B \big|A(\nabla h)(x) - \langle A(\nabla h)\rangle_B\big|\,dx$$

for some $\alpha, \beta > 0$ and all $\theta \in (0, \frac{1}{2})$.

Two cases

Non-degenerate case:

$$\fint_B |V(\nabla u) - \langle V(\nabla u)\rangle_B|^2\,dx \ll \fint_B |V(\nabla u)|^2\,dx$$

Compare with linear system
$$\begin{aligned} -\operatorname{div}((DA)(Q)\nabla z) &= 0 \qquad \text{on } B,\\ z &= u \qquad \text{on } \partial B. \end{aligned}$$

Degenerate case:

$$\fint_B |V(\nabla u) - \langle V(\nabla u)\rangle_B|^2\,dx \approx \fint_B |V(\nabla u)|^2\,dx.$$

Compare with p-harmonic system
$$\begin{aligned} -\operatorname{div}(A(\nabla z)) &= 0 \qquad \text{on } B,\\ z &= u \qquad \text{on } \partial B. \end{aligned}$$

In both cases: Decay estimate of z transfers to u.

Transfer the decay

As usual we transfer the decay of z to u by

$$
\begin{aligned}
&\fint_{\theta B} |A(\nabla u) - \langle A(\nabla u)\rangle_{\theta B}|\, dx \\
&\qquad \lesssim \fint_{\theta B} |A(\nabla z) - \langle A(\nabla z)\rangle_{\theta B}|\, dx + \fint_{\theta B} |A(\nabla u) - A(\nabla z)|\, dx \\
&\qquad \lesssim \theta^{\alpha} \fint_{B} |A(\nabla z) - \langle A(\nabla z)\rangle_{B}|\, dx + \theta^{-n} \fint_{B} |A(\nabla u) - A(\nabla z)|\, dx \\
&\qquad \lesssim \theta^{\alpha} \fint_{B} |A(\nabla u) - \langle A(\nabla u)\rangle_{B}|\, dx + \theta^{-n} \fint_{B} |A(\nabla u) - A(\nabla z)|\, dx
\end{aligned}
$$

Estimate last term by data using a comparison estimate.

Degenerate case:

$$\fint_B |V(\nabla u) - \langle V(\nabla u)\rangle_B|^2\,dx \approx \fint_B |V(\nabla u)|^2\,dx.$$

Now in any estimate involving $\varphi_{p',|A(Q)|}$ we can change to $\varphi_{p'}$ using:

$$\begin{aligned}\varphi_{p',|A(Q)|}(t) &\leq c_\delta\, t^{p'} + \delta\,|Q|^p,\\ t^{p'} &\leq c_\delta\,\varphi_{p',|A(Q)|}(t) + \delta\,|Q|^p.\end{aligned}$$

Then for $V(Q) = \langle V(\nabla u)\rangle_B$ the extra term is bounded by oscillation:

$$|Q|^p \lesssim |\langle V(\nabla u)\rangle_B|^2 \lesssim \fint_B |V(\nabla u) - \langle V(\nabla u)\rangle_B|^2\,dx.$$

Non-degenerate case:

$$\fint_B \left|V(\nabla u) - \langle V(\nabla u)\rangle_B\right|^2 dx \ll \fint_B |V(\nabla u)|^2 dx$$

To get rid of the shifts we use

$$\left(\fint_B |g|^{\min\{2,p'\}} dx\right)^{\frac{1}{\min\{2,p'\}}} \lesssim \varphi^{-1}_{p',|A(Q)|}\left(\int_B \varphi_{p',|A(Q)|}(g)\, dx\right).$$

Finally:

Theorem [Breit, Cianchi, Diening, Kuusi, Schwarzacher '15]

$$M^\sharp_{\min\{2,p'\}}(A(\nabla u)) \lesssim M^\sharp_{p'}(F).$$

Conclusion

Let $-\mathrm{div}(A(\nabla u)) = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mathrm{div}(F)$.

1. Pointwise estimates in terms of maximal operators

$$M^\sharp(A(\nabla u)) \lesssim M^\sharp_{p'}(F),$$
$$M^\sharp_\omega(A(\nabla u)) \lesssim M^\sharp_{p',\omega}(F).$$

2. Regularity of F transfers to $A(\nabla u)$ as in the linear case.
   $\Rightarrow$ „Nonlinear Calderón-Zygmund theory“
   Examples: L^q, BMO, $C^{0,\alpha}$, $L^{q,s}$, $L^{\exp}$, ...
3. Potential type estimate

$$|A(\nabla u(x))| \lesssim \int_0^\infty \bigg(⨍_{B_r(x)} \bigg| \frac{|F - \langle F\rangle_{B_r(x)}|}{r} \bigg|^{p'} dy \bigg)^{\frac{1}{p'}} \frac{dr}{r}.$$

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Thank you for your attention!