Norm estimates for the Hardy operator in terms of B_p weights

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Índex

Introduction

Sharp dependence of the norm of operators in harmonic analysis, acting in weighted spaces.

- Hardy-Littlewood maximal operator (Buckley, Hytönen, Pérez, Lerner).
- Hilbert and Riesz transforms (Petermichl).
- Calderón-Zygmund operators (Hytönen).

For the classical Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where the supremum is taken over all cubes $Q \in \mathbb{R}^n$ containing x. Let w be a weight, that is, a positive locally integrable function, and, for a given measurable set E, let $u(E) = \int_E u(x) \, dx$, and for $p > 1$, let us define $\sigma = u^{-1/(p-1)}$.

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Introduction

We say that u satisfies the A_p condition if

$$[u]_{A_p} = \sup_Q \frac{u(Q)\sigma(Q)^{p-1}}{|Q|^p} < \infty.$$

For $p = 1$, the class A_1 of weights is characterized as those for which, for all cubes Q,

$$\frac{u(Q)}{|Q|} \leq C \inf_{x \in Q} u(x).$$

and the best constant C in the above inequality it is denoted by the $[u]_{A_1}$ constant.

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$$\|M\|_{L^p(u)} \lesssim [u]_{A_p}^{1/(p-1)},$$

and the exponent $1/(p-1)$ is the best possible.

- The sharp constant in the weak-type boundedness of M on $L^p(u)$ was also studied by Buckley and it was obtained that, for $1 \le p < \infty$

$$\|M\|_{L^p(u) \longrightarrow L^{p,\infty}(u)} \simeq [u]_{A_p}^{1/p}.$$

- As a consequence of the two previous facts, using the trivial embedding $L^p(u) \hookrightarrow L^{p,\infty}(u)$, we obtain

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Let us consider $Sf(t) = t^{-1} \int_0^t f(s)\, ds$, the classical Hardy operator, since $(Mf)^* \approx S(f^*)$, where f^* denotes the classical decreasing rearrangement with respect the Lebesgue measure. The action of the maximal operator M on classical Lorentz spaces with respect some weight w, $0 < p < \infty$,

$$\Lambda^p(w) := \left\{ f;\ \|f\|_{\Lambda^p(w)} := \left(\int_0^\infty (f^*(t))^p\ w(t)\ dt \right)^{1/p} < +\infty \right\},$$

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Introduction

For $p > 0$, we recall here that a positive and measurable function $w \in B_p$ if there exists a positive constant $C > 0$ such that

$$r^p \int_r^\infty \frac{w(x)}{x^p}\, dx \le C \int_0^r w(x)\, dx.$$

Weights in the B_p class are exactly those for which the maximal operator M is bounded in the Lorentz space $\Lambda^p(w)$ (Ariño and B. Muckenhoupt, 1990). Testing the boundedness of S on characteristic functions, $f(x) = \chi_{(0,r)}(x)$, we obtain the following in terms of this optimal constant C

$$\int_0^\infty \left(\int_0^r \frac{\chi_{(0,x)}(t)}{x}\, dt \right)^p w(x)\, dx \le (1 + C) \int_0^r w(x)\, dx.$$

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For this reason it is natural to express the dependence on the B_p condition of the weight in terms of the quantity

$$[w]_{B_p} := 1 + \sup_{r>0} \frac{r^p \displaystyle\int_r^\infty \frac{w(x)}{x^p}\, dx}{\displaystyle\int_0^r w(x)\, dx}.$$

We will consider also the weak-type Lorentz spaces $\Lambda^{p,\infty}(w)$, $0 < p < \infty$.

$$\Lambda^{p,\infty}(w) = \left\{ f;\ \|f\|_{\Lambda^{p,\infty}(w)} = \sup_{t>0} f^*(t) \left(\int_0^t w(s)\, ds \right)^{1/p} < \infty \right\},$$

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Introduction

Let us consider the action of the Hardy operator S in the following three cases:

- (SS) $S : L^p_{dec}(w) \longrightarrow L^p(w)$, $0 < p < \infty$.
- (WW) $S : L^{p,\infty}_{dec}(w) \longrightarrow L^{p,\infty}(w)$, $0 < p < \infty$.
- (SW) $S : L^p_{dec}(w) \longrightarrow L^{p,\infty}(w)$, $1 < p < \infty$.

We will use the following notation: $\|S\|_{p,w}$,$\|S\|_{(p,\infty),w}$ and $\|S\|^*_{p,w}$, respectively, for denoting the norm $\|S\|$ in each of the three cases described above.

Introduction

There is no weight for which $S : L^{p,\infty}_{dec}(w) \longrightarrow L^p(w)$ is bounded.

Otherwise, the embedding $S : L^{p,\infty}_{dec}(w) \hookrightarrow L^p(w)$ would be continuous and this is equivalent to

$$\int_0^\infty \frac{w(t)}{W(t)}\, dt < \infty,$$

which is false.

Introduction

The explicit expression for $\|S\|_{p,w}$ in case (SS), when $p > 1$, was completely solved by E. Sawyer (1981):

$$\|S\|_{p,w} \simeq 1 + \sup_{t>0} \left(\int_t^\infty \frac{w(x)}{x^p}\, dx \right)^{1/p} \left(\int_0^t \left(\frac{W(x)}{x} \right)^{-p'} w(x)\, dx \right)^{1/p'}.$$

For $0 < p \leq 1$, it can be shown that $\|S\|_{p,w} = [w]_{B_p}^{1/p}$ (Carro, Soria, 1993). Thus, whenever $0 < p < \infty$, we have that $\|S\|_{p,w} < \infty$ if and only $w \in B_p$.

The estimate for $\|S\|_{(p,\infty),w}$ in (WW) was characterized by Soria (1998), for $0 < p < \infty$:

$$\|S\|_{(p,\infty),w} = \sup_{t>0} \frac{1}{t} \left(\int_0^t \frac{1}{W^{1/p}(s)}\, ds \right) W^{1/p}(t),$$

where it was also proved that $\|S\|_{(p,\infty),w} < \infty$ if and only $w \in B_p$.

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Introduction

Concerning the explicit expression for $\|S\|^*_{p,w}$ in (SW), when $p>1$, we observe that, as a consequence of the work of Sawyer,

$$\|S\|^*_{p,w} = \sup_{f \text{ dec}} \frac{\|Sf\|_{L^{p,\infty}(w)}}{\|f\|_{L^p(w)}} = \sup_{t>0} \sup_{f \text{ dec}} \frac{\int_0^\infty f(x)\chi_{(0,t)}(x)\,dx}{\left(\int_0^\infty f^p(s)w(s)\,ds\right)^{1/p}} \frac{W^{1/p}(t)}{t}$$

$$\simeq \sup_{t>0} \left(\int_0^t x^{p'-1} W(x)^{1-p'}\,dx\right)^{1/p'} \frac{W^{1/p}(t)}{t},$$

and this is again equivalent to the fact that $w \in B_p$.

Introduction

We remark that for $0 < p \leq 1$, the necessary and sufficient condition for the boundedness of $S : L^p_{dec}(w) \longrightarrow L^{p,\infty}(w)$ is (Carro, J. Soria [JFA, 1993]; Carro, García del Amo, J. Soria [PAMS, 1996]) that the primitive of the weight $W(t) = \int_0^t w(x)\, dx$ is a p-quasi concave function, that is, for all $0 < s \leq r < \infty$,

$$\frac{W(r)}{r^p} \leq C \frac{W(s)}{s^p}.$$

In fact,

$$\|S\|^*_{p,w} = \sup_{0<s\leq r<\infty} \frac{s}{r} \left(\frac{W(r)}{W(s)} \right)^{1/p},$$

and $\|S\|^*_{p,w} < \infty$ is a weaker condition than $w \in B_p$, $0 < p \leq 1$.

Similarly to what is done for the Hardy-Littlewood maximal operator M, our main interest now is to study good bounds for the exponents α and β, so that the following inequalities hold

$$[w]_{B_p}^{\alpha} \lesssim \|S\| \lesssim [w]_{B_p}^{\beta}, \tag{1}$$

where $\|S\|$ denotes any of the three norms in (SS), (WW), or (SW). The optimal bounds for the exponents α and β in (1) can be determined as follows:

$$\alpha_p := \sup \left\{ \alpha \geq 0 : \inf_{w \in B_p} \frac{\|S\|}{[w]_{B_p}^{\alpha}} > 0 \right\},$$

and

$$\beta_p := \inf \left\{ \beta \geq 0 : \sup_{w \in B_p} \frac{\|S\|}{[w]_{B_p}^{\beta}} < \infty \right\}.$$

Main results

Although not explicitly written in a quantitative form, the dependence on the B_p constant of the norm of the Hardy operator $S : L^p_{\text{dec}}(w) \longrightarrow L^p(w)$, which corresponds to the case (SS), are contained in the following theorem:

THEOREM.([Carro, Soria] Can. J. Math.)

Let $0 < p < \infty$ and $w \in B_p$. Then,

(a) If $0 < p \le 1$,

$$\|S\|_{p,w} = [w]_{B_p}^{1/p}.$$

Hence, with $\|S\| = \|S\|_{p,w}$, we obtain the optimal estimates $\alpha_p = \beta_p = 1/p$.

(b) If $p > 1$,

$$[w]_{B_p}^{1/p} \le \|S\|_{p,w} \le [w]_{B_p}.$$

Moreover, we obtain the estimates $1/p \le \alpha_p \le 1 = \beta_p$.

Main results

- Considering characteristic functions $f = \chi_{(0,r)}$, we easily obtain that $\|S\|_{p,w} \geq [w]_{B_p}^{1/p}$.
- To obtain the other estimate we must write the p-power of the weighted norm in terms of the distribution function and use optimal estimates of some embeddings between Lorentz spaces.
- The optimality of the exponent in the right hand side follows by considering the family of power weights $w_\alpha(x) = x^\alpha$, $-1 < \alpha < p-1$, then

$$\|S\|_{L^p(w_\alpha)} = \frac{p}{p-\alpha-1} = [w_\alpha]_{B_p}.$$

We observe that since $\lim_{\alpha \to (p-1)^-} [w_\alpha] = \infty$, the sharpness in the upper bound of the statement holds.

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Main results

The proofs of the estimates for the norms $\|S\|_{(p,\infty),w}$ and $\|S\|^*_{p,w}$, the cases (WW) and (SW), are based in the following improvement of a result due to Y. Sagher, (see also Gogatishvili, Kufner and Persson for some related estimates):

Proposition.

Let m be a positive function and $\varepsilon > 0$. Then,

(i) The existence of two positive constants A and B such that, for every $r > 0$

$$Am(r) \leq \int_0^r \frac{m(s)}{s}\,ds \leq Bm(r),$$

implies

$$\frac{A^{\varepsilon+1}}{\varepsilon B^{\varepsilon}}\frac{1}{m^{\varepsilon}(r)} \leq \int_r^{\infty} \frac{1}{m^{\varepsilon}(s)}\frac{ds}{s} \leq \frac{B^{\varepsilon+1}}{\varepsilon A^{\varepsilon}}\frac{1}{m^{\varepsilon}(r)}.$$

Main results

(ii) Conversely, the existence of two positive constants C and D such that, for every $r > 0$

$$\frac{C}{m(r)} \leq \int_r^\infty \frac{1}{m(s)} \frac{ds}{s} \leq \frac{D}{m(r)},$$

implies

$$\frac{C^{\varepsilon+1}}{\varepsilon D^\varepsilon} m^\varepsilon(r) \leq \int_0^r \frac{m^\varepsilon(s)}{s} ds \leq \frac{D^{\varepsilon+1}}{\varepsilon C^\varepsilon} m^\varepsilon(r).$$

Main results

From the corresponding estimates for $\|S\|_{(p,\infty),w}$ in (WW) due to Soria, and using Sagher's generalization result to appropriate functions,

Theorem.

If $0 < p < \infty$ and w is weight in B_p, then

$$[w]_{B_p}^{1/(p+1)} \leq \|S\|_{(p,\infty),w} \leq [w]_{B_p}^{(p+1)/p}.$$

Moreover, we obtain the following estimates

$$\frac{1}{(p+1)} \leq \alpha_p \leq 1 \leq \beta_p \leq \frac{p+1}{p}.$$

Main results

Let us observe that for power weights in the B_p class, $w_\alpha(t) = t^\alpha$, $-1 < \alpha < p - 1$, we can explicitly calculate

$$\|S\|_{(p,\infty),w_\alpha} = \frac{p}{p - \alpha - 1} = [w_\alpha]_{B_p}.$$

Therefore, we get

$$\frac{1}{(p+1)} \leq \alpha_p \leq 1 \leq \beta_p \leq \frac{p+1}{p}.$$

Main results

We now study the estimates for $\|S\|_{p,w}^*$ in (SW).

Theorem.

Let $p > 1$ and $w \in B_p$. Then,

$$[w]_{B_p}^{1/(pp')} \lesssim \|S\|_{p,w}^* \lesssim [w]_{B_p}.$$

Moreover, we obtain the following estimates

$$\frac{1}{pp'} \leq \alpha_p \leq \frac{1}{p'} \leq \beta_p \leq 1.$$

Main results

It follows from the idea and techniques as the case (WW) but for the estimate due to Sawyer (Indiana Univ. Math. J, 1981) valid for $p > 1$

$$\|S\|^*_{p,w} \simeq \sup_{t>0} \left(\int_0^t x^{p'-1} W(x)^{1-p'}\, dx \right)^{1/p'} \frac{W^{1/p}(t)}{t}.$$

Also, for power weights $w_\alpha(t) = t^\alpha$, $-1 < \alpha < p-1$, we can explicitly calculate the above expression and obtain

$$\|S\|^*_{p,w_\alpha} \simeq \left(\frac{p-1}{p-\alpha-1} \right)^{1/p'} = \left(\frac{1}{p'} \right)^{1/p'} [w_\alpha]^{1/p'}_{B_p}.$$

Therefore, we get

$$\frac{1}{pp'} \leq \alpha_p \leq \frac{1}{p'} \leq \beta_p \leq 1.$$

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